A practical procedure to estimate the shape parameter in the generalized Gaussian distribution

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Abstract. We propose a method to estimate the shape parameter p in the generalized Gaussian distribution. Our estimator is an explicit approximate solution to the trascendental estimator obtained by the method of moments. An estimator for p, based on the method of moments, does not always exist, however we show that it is possible to find such an estimator with high probability for most of the practical situations. A numeric-analytical procedure to obtain the confidence intervals for p is also presented. We illustrate our procedures on data obtained from the different subbands of the audio MP3 encoder.

Key words: generalized Gaussian distribution, method of moments, generalized Gaussian ratio function (ggrf), sampled generalized Gaussian ratio function (sggrf), Gurland's inequality, confidence intervals.

1 Introduction

The Gaussian distribution is a typical model for signals and noise in many applications in science and engineering. However, there are some applications where this Gaussian assumption departs from the actual random behavior. For instance, the samples of a speech signal are modeled by a Laplacian distribution, and the generalized Gaussian distribution has been proposed for modeling atmospheric noise, subband encoding of audio and video signals [10], impulsive noise, direction of arrival, independent component analysis [1], blind signal separation [12], GARCH [6], etc..

The generalized Gaussian (GG) distribution can be parametrized in such a manner that its mean (μ) and variance (σ^2) coincide with the Gaussian distribution. [1] Additionally to the mean and variance, the GG has the shape parameter p, which is a measure of the peakedness of the distribution, however, it seems that there is not a closed-form expression for estimating p. The parameter p determines the shape of the distribution, e.g., the Gaussian distribution is obtained for ($p = 2$), the Laplacian distribution for ($p = 1$), and by making $p \to 0$ we can obtain a distribution close to the uniform distribution. In most of the applications the mean can be considered as zero, then we will be focused on estimating the shape parameter of the GG distribution with two parameter, i.e., $\mu = 0$.

Varanasi and Aazhang [11] discuss parameter estimation for the GG by using the methods of maximum likelihood and moments. Rodríguez-Dagnino and León-García [9] present a closed-form estimator based on the Gurland's inequality. There are some computational difficulties regarding the mathematical expressions presented in [11], mostly related to the gamma function, whereas the approximation proposed in [9] is only well-behaved on the range $0.3 < p < 3$, which is important for subband encoding of video signals, and some related applications. However, the interval is not wide enough to cover most of the cases. In particular, in this work we have obtained approximations to cover the range $0.18 < p < 1.32$

[1] See Appendix A.

in a more precise manner (see Section 5).

In this paper, we propose a simple method to estimate p, which gives explicit expressions for estimating the shape parameter. The method follows the ideas proposed by López [5], however we have extended the range for estimating p, and we have built corresponding confidence intervals with a specified covering probability for the shape parameter.

2 Generalized Gaussian Distribution

A random variable X is distributed as generalized Gaussian if its probability density function (*pdf*) is given by

$$\mathrm{gg}\left(x;\mu,\sigma,p\right)=\frac{1}{2\Gamma\left(1+1/p\right)\mathrm{A}\left(p,\sigma\right)}e^{-\left|\frac{x-\mu}{\mathrm{A}(p,\sigma)}\right|^{p}},\quad x\in\mathbb{R} \tag{1}$$

where $\mu\in\mathbb{R},\ p,\sigma>0$ y $\mathrm{A}\left(p,\sigma\right)=\left[\frac{\sigma^{2}\Gamma(1/p)}{\Gamma(3/p)}\right]^{1/2}$. The parameter μ is the mean, the function $\mathrm{A}\left(p,\sigma\right)$ is an scaling factor which allows that $\mathrm{Var}\left(X\right)=\sigma^{2}$, and p is the shape parameter. As we notice above, when $p=1$, the GG corresponds to a Laplacian or double exponential distribution, $p=2$ corresponds to a Gaussian distribution, whereas in the limiting cases $p\to+\infty$ the *pdf* in equation (1) converges to a uniform distribution in $\left(\mu-\sqrt{3}\sigma,\mu+\sqrt{3}\sigma\right)$, and when $p\to0+$ the distribution becomes a degenerate one in $x=\mu$ (see appendix A).

We will use the following notation: $X\sim\mathrm{GG}\left(\mu,\sigma,p\right)$ to denote that X is a random variable with *pdf* as in equation 1, and we will denote $\mathrm{GG}\left(\sigma,p\right)=\mathrm{GG}\left(0,\sigma,p\right)$.

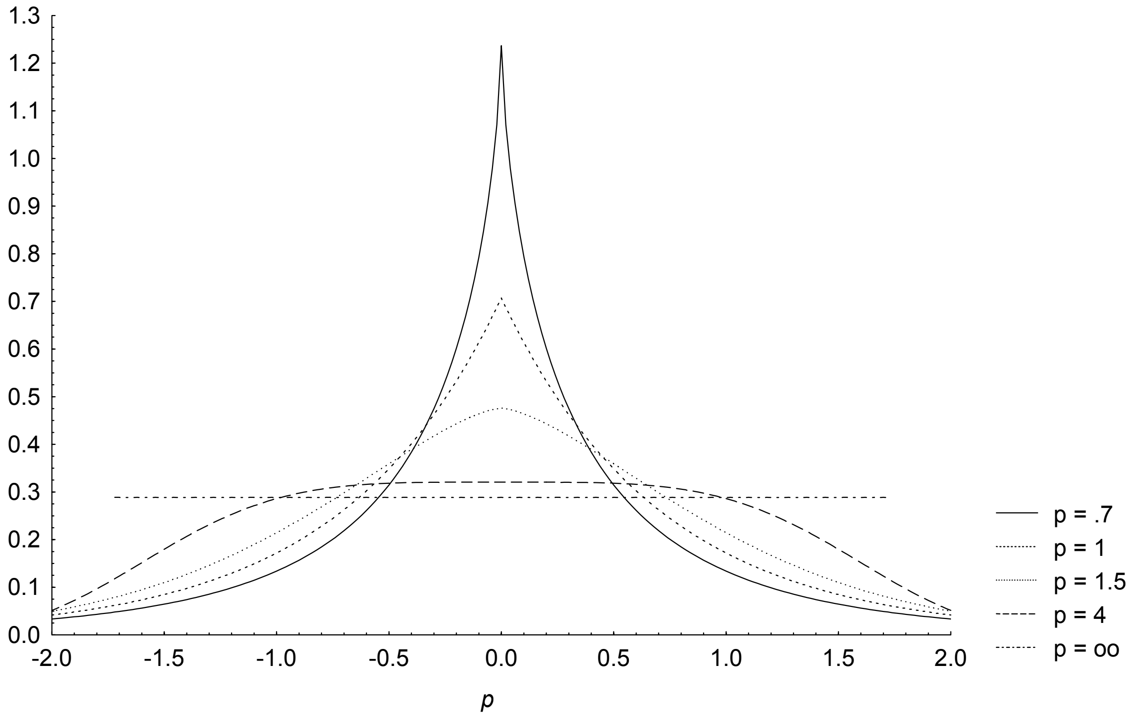

Figure 1: *Generalized Gaussian* pdf*'s for different values of* p. *From the top to the bottom:* $p = 0.7, 1, 1.5, 2, 4.$

The GG distribution is symmetric with respect to μ, hence the odd central moments are zero, i.e., $E(X-\mu)^r = 0, \quad r = 1, 3, 5, ...$ The even central moments can be obtained from the absolute central moments, which are given by

$$E|X-\mu|^r = \left[\frac{\sigma^2\Gamma(1/p)}{\Gamma(3/p)}\right]^{r/2} \frac{\Gamma\left(\frac{r+1}{p}\right)}{\Gamma(1/p)}. \tag{2}$$

In particular, the variance of X is

$$\text{Var}(X) = E(X-EX)^2 = E(X-\mu)^2 = EY^2 = \sigma^2.$$

3 Existence of the moment estimator

Varanasi & Aazhang (1989) present three methods to estimate the parameters of the GG(μ, σ, p) distribution, namely the maximum likelihood estimator (MLE), the method of moments estimator(MME), and a combination of both of them. In this work, we will be focused on estimating only the shape parameter by using the method of moments. In order to estimate the shape parameter Varanasi & Aazhang (1989) suggests the use of any moments higher or equal to 4, and it is necessary to solve the following equation

$$\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^r = \frac{[A(\tilde{p}_n)]^r\,\Gamma\left(\frac{r+1}{\tilde{p}_n}\right)}{\Gamma\left(\frac{1}{\tilde{p}_n}\right)}, \quad \tilde{p}_n>0, \quad r\geq 4, \quad r=2m,\ m\in\mathbb{N},\ m\geq 2.$$

Hence, the MME $\tilde{p}_n$, of p, is given by the value satisfying the following equation

$$\frac{\left[\Gamma\left(\frac{1}{\tilde{p}_n}\right)\right]^{r/2-1}\Gamma\left(\frac{r+1}{\tilde{p}_n}\right)}{\Gamma\left(\frac{1}{\tilde{p}_n}\right)} = \frac{\frac{1}{n}\sum\limits_{i=1}^{n}(x_i-\bar{x})^r}{\left[\frac{1}{n}\sum\limits_{i=1}^{n}(x_i-\bar{x})^2\right]^{r/2}} \tag{3}$$

However, Varanasi & Aazhang (1989) did not mention that the equation (3), may not have a solution. This is so because for each $r\geq 1$, the leftmost function is a decreasing function in $(0,\infty)$, and it satisfies the following limits:

$$\lim_{p\to 0+}\frac{\left[\Gamma\left(\frac{1}{p}\right)\right]^{r/2-1}\Gamma\left(\frac{r+1}{p}\right)}{\left[\Gamma\left(\frac{3}{p}\right)\right]^{r/2}}=\infty, \quad \lim_{p\to\infty}\frac{\left[\Gamma\left(\frac{1}{p}\right)\right]^{r/2-1}\Gamma\left(\frac{r+1}{p}\right)}{\left[\Gamma\left(\frac{3}{p}\right)\right]^{r/2}}=\frac{3^{\frac{r}{2}}}{1+r}. \tag{4}$$

The right-most limit of (4) is obtained by applying (E.4), whereas the right-most function of equation (3) satisfies the following inequality (see Appendix C),

$$1\leq\frac{\frac{1}{n}\sum\limits_{i=1}^{n}(x_i-\bar{x})^r}{\left[\frac{1}{n}\sum\limits_{i=1}^{n}(x_i-\bar{x})^2\right]^{r/2}}\leq n^{\frac{r}{2}-1}$$

The inequality is valid only when $r\geq 2$ and $r=2m$, $m\geq 1$. Then, when

$$1\leq\frac{\frac{1}{n}\sum\limits_{i=1}^{n}(x_i-\bar{x})^r}{\left[\frac{1}{n}\sum\limits_{i=1}^{n}(x_i-\bar{x})^2\right]^{r/2}}\leq\frac{3^{\frac{r}{2}}}{1+r}, \tag{5}$$

the equation (3) does not have any solution.

Let us consider $r = 4$, then $\tilde{p}_n$, is given by the solution to

$$\frac{\Gamma\left(\frac{1}{\tilde{p}_n}\right)\Gamma\left(\frac{5}{\tilde{p}_n}\right)}{\Gamma^2\left(\frac{3}{\tilde{p}_n}\right)} = \frac{\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^4}{\left(\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2\right)^2}. \tag{6}$$

For example, for the data in Table F.3 we have that

$$\bar{x} = \frac{1}{25}\sum_{i=1}^{25} x_i = 0.27353, \qquad \frac{\frac{1}{25}\sum_{i=1}^{25}(x_i-0.27353)^4}{\left(\frac{1}{25}\sum_{i=1}^{25}(x_i-0.27353)^2\right)^2} = 1.63664,$$

and for the data of Table F.4 we can obtain

$$\bar{z} = \frac{1}{25}\sum_{i=1}^{25} z_i = 0.032141, \quad \frac{\frac{1}{25}\sum_{i=1}^{25}(z_i-0.03214\,1)^4}{\left(\frac{1}{25}\sum_{i=1}^{25}(z_i-0.03214\,1)^2\right)^2} = 1.5207$$

Since

$$\frac{3^{\frac{4}{2}}}{1+4} = \frac{9}{5} = 1.8 < \frac{\Gamma\left(\frac{1}{\tilde{p}_n}\right)\Gamma\left(\frac{5}{\tilde{p}_n}\right)}{\Gamma^2\left(\frac{3}{\tilde{p}_n}\right)},$$

we can conclude that equation (6) has not any solution. It means, it does not exist any r such that we can solve equation (3), for the data in Tables F.3 and F.4, therefore the MME for p does not exist.

The probability for the existence of the MME of p is determined by the probability of the event defined by equation (5). This probability depends only on n and p, and it does not depend on either μ nor σ. Based on the consistency of the MME we have that if r is fixed and n is large, then the probability of the existence of the MME increases as well. Furthermore, as it will be illustrated in the next section, the probability of existence depends strongly on p. We should also observe that for p large it is necessary to increase the sample size in order to ensure, with a high probability, the existence of the MME.

In order to estimate p, we assume μ known and σ unknown. We consider the first two absolute moments, as they are defined in (2), to obtain the MME of σ and p. It means,

$$\frac{1}{n}\sum_{i=1}^{n}\left|X_i-\mu\right|^k = E\left|X-\mu\right|^k$$

From the equation (A.16) we have

$$E\left|X\right| = \sigma\sqrt{M\left(p\right)}, \quad \text{and} \quad EX^2=\sigma^2, \tag{7}$$

where

$$M\left(p\right) = \frac{\left(E\left|X\right|\right)^2}{EX^2} = \frac{\Gamma^2\left(\frac{2}{p}\right)}{\Gamma\left(\frac{1}{p}\right)\Gamma\left(\frac{3}{p}\right)}, \tag{8}$$

the reciprocal function of $M\left(p\right)$ is known as *generalized Gaussian function ratio (ggfr).*

Hence, by using equations (7) we can obtain the MME for σ and p after solving the following equations:

$$\frac{1}{n}\sum_{i=1}^{n}\left|X_i\right| = \sigma\sqrt{M\left(p\right)} \quad \text{and} \quad \frac{1}{n}\sum_{i=1}^{n}\left|X_i\right|^2=\sigma^2. \tag{9}$$

By obtaining σ and $M\left(p\right)$ from the equations (9) we can have

$$\bar{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}\left|X_i-\mu\right|^2 \quad \text{and} \quad M\left(\bar{p}\right) = \bar{\mathrm{M}}\left(X\right) = \frac{\left(\frac{1}{n}\sum_{i=1}^{n}\left|X_i-\mu\right|\right)^2}{\frac{1}{n}\sum_{i=1}^{n}\left|X_i-\mu\right|^2}. \tag{10}$$

The reciprocal function of $\bar{\mathrm{M}}\left(X\right)$ is an statistic that we name as *sampled generalized Gaussian function ratio (sggfr).*

Now, in order to solve (9) we must solve the following equation

$$M\left(\bar{p}\right) = \bar{\mathrm{M}}\left(X\right),$$

which not always has a solution since the range of the function [2] $\mathrm{M}\left(p\right)$ is $\left(0,\frac{3}{4}\right)$, and sggfr satisfies[3] $\frac{1}{n}\leq\bar{\mathrm{M}}\left(X\right)\leq 1$, which shows that if $\frac{3}{4}<\bar{\mathrm{M}}\left(X\right)\leq 1$, it is not possible to solve the equations in (9).

[2] See Appendix E

[3] See Appendix C

When $\frac{1}{n} < \bar{\mathrm{M}}(X) < \frac{3}{4}$, then the solution of the equations (9) is given by

$$\bar{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} |X_i - \mu|^2 \quad \text{and} \quad M(\bar{p}) = \bar{\mathrm{M}}(X), \tag{11}$$

Thus, the MME for p is given by

$$\bar{p} = M^{-1}\left[\bar{\mathrm{M}}(X)\right], \tag{12}$$

where $M^{-1}(\cdot)$ represents the inverse function of $M(\cdot)$.

In most of the cases the method of moments produces consistent estimators. In our case, since the GG distribution has all positive moments, we have that $\frac{1}{n}\sum_{i=1}^{n}|X_i - \mu|^2$ and $\frac{1}{n}\sum_{i=1}^{n}|X_i - \mu|$ converge in probability to σ^2 and $\sigma\sqrt{M(p)}$ respectively. Therefore, the *sggfr* is a consistent estimator of $M(p)$.

We should make the following remarks:

1. The probability distribution of the MME for p does not depend on σ.

2. Since $\bar{\mathrm{M}}(x) \geq \frac{1}{n}$, then it is not possible that p takes values inside $\mathrm{M}^{-1}\left(\left[0, \frac{1}{n}\right]\right) = \left[0, \mathrm{M}^{-1}\left(\frac{1}{n}\right)\right]$.

3. For large samples and p small ($p < 5$), which is common in applications, the event $\frac{3}{4} \leq \bar{\mathrm{M}}(x) \leq 1$ occurs with very small probability (see Figure 2). We can interpret this probability as *the probability of existence of the MME of p*. Then, if we observe an event x such that $\frac{3}{4} \leq \bar{\mathrm{M}}(x) \leq 1$ and n is large, it will indicate either that p is very large or that the actual data distribution departs significantly from the generalized Gaussian. [4]

4. It can be observed, in Figure 2, the minimum sample size needed such that the probability of the event $\bar{\mathrm{M}}(x) \geq \frac{3}{4}$ is smaller than or equal to 0.05. It means, we have the minimum sample size such that the probability that the MME for p exists is 0.95. We should observe that the minimum n^* is an increasing function of p. For instance, the minimum

[4] In Appendix E we present some alternative distributions, such as lognormal and generalized gamma, that might be useful for some applications. Our interest on finding distributions that satisfy sggfr $\geq \frac{3}{4}$, is on estimating the corresponding parameters by following a similar procedure as we do for the GG.

sample size such that $\Pr\left(\bar{\mathrm{M}}\left(x\right) \geq \frac{3}{4}\right)$ is close to zero, when p takes values in the interval $[0.3, 3]$, (which are typical in many applications), is $n = 61$. Similarly, when $p \leq 5$ then the minimum sample size is 216. Any of the n values obtained above is typically exceeded in applications.

5. When the sample size is large enough, we have that $\frac{1}{n}$ is small, which allow us to obtain estimated values for p close to zero.

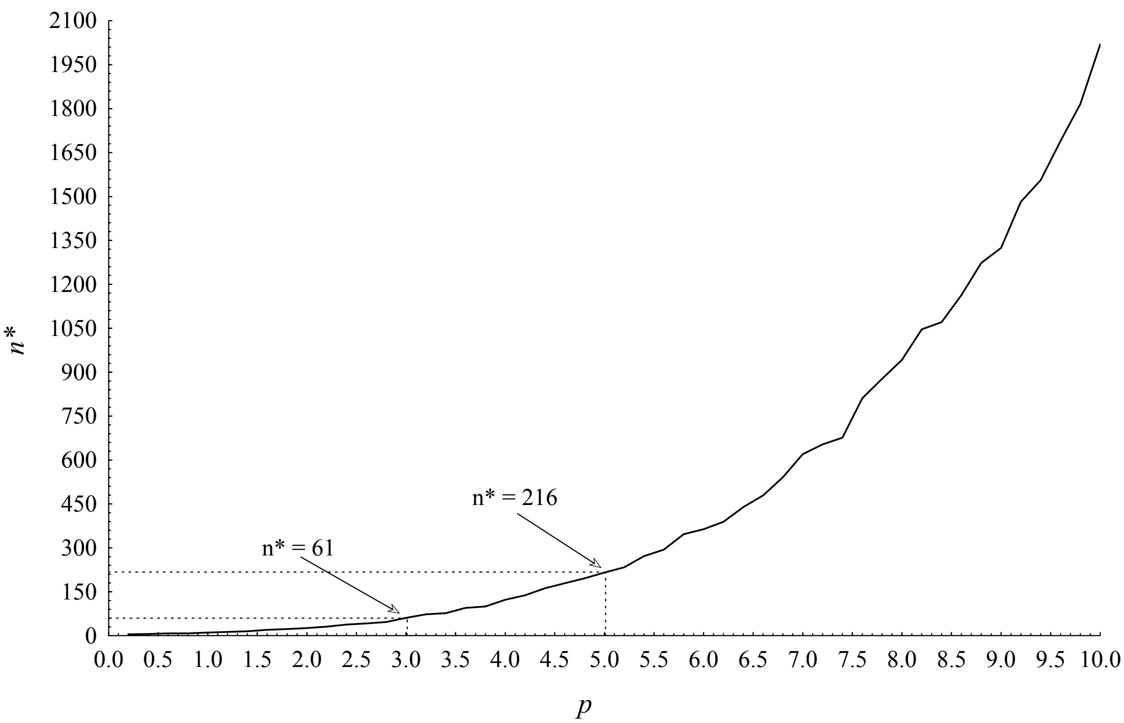

Figure 2.. Values of p versus $n^* = \min\left\{n : \Pr\left(\bar{M}\left(X\right) \geq \frac{3}{4}; p\right) > 0.05\right\}, \; p \in (0, 10)$.

6. If we consider μ unknown, the MME $\bar{\mu}$, of μ, is $\bar{\mu} = \bar{X}$. Then the MME of σ and p is the same as in (11) and (12) with μ substituted by $\bar{X}$.

7. We should remark that the existence problems for the MME for p are the same in either case μ unknown or not. In this work we assume μ known and equal to zero.

8. For the data presented in Tables F.3 y F.4 we have that $\bar{x} = -0.27353$ and $\bar{z} =$

0.032141 and

$$\frac{\left(\frac{1}{25}\sum_{i=1}^{25}|x_i|\right)^2}{\frac{1}{25}\sum_{i=1}^{25}x_i^2}=0.7786,\quad \frac{\left(\frac{1}{25}\sum_{i=1}^{25}|x_i-0.27353|\right)^2}{\frac{1}{25}\sum_{i=1}^{25}(x_i-0.27353)^2}=0.8151$$

$$\frac{\left(\frac{1}{25}\sum_{i=1}^{25}|z_i|\right)^2}{\frac{1}{25}\sum_{i=1}^{25}z_i^2}=0.8402,\quad \frac{\left(\frac{1}{25}\sum_{i=1}^{25}|z_i-0.032141|\right)^2}{\frac{1}{25}\sum_{i=1}^{25}(z_i-0.032141)^2}=0.8517,$$

and the solution does not exist if we consider either μ known or unknown.

3.1 Approximation of $\mathrm{M}(p)$

It seems to be clear that the function $\mathrm{M}(p)$ cannot be inverted in an explicit form. From this prospective, we propose an approximation such that it can be inverted and close enough to the actual function in a range of values of p useful in applications.

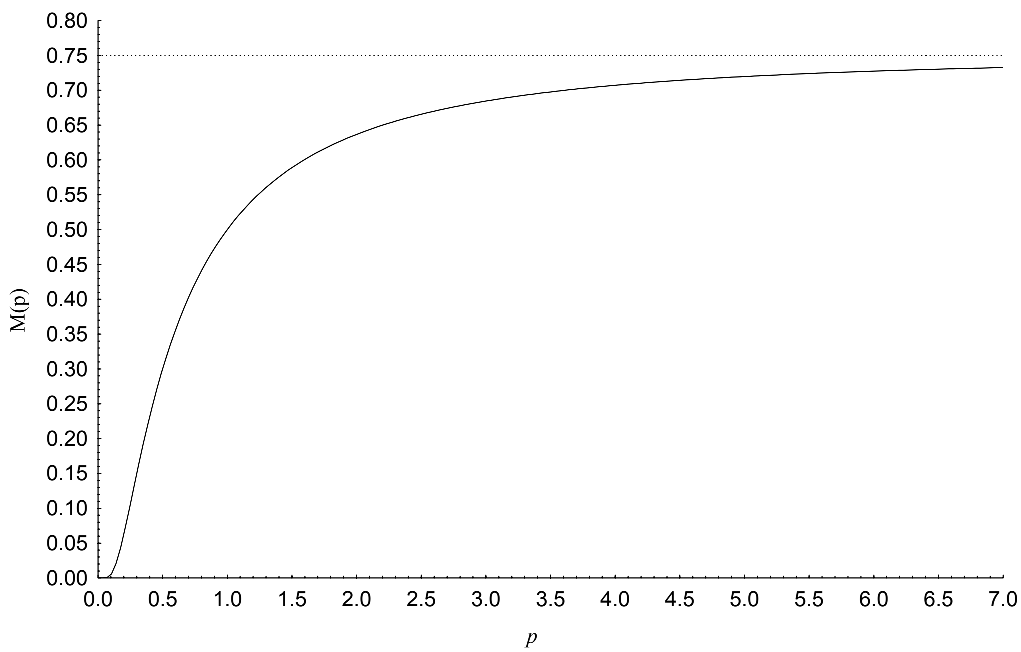

Figure 3. Behavior of the function $\mathrm{M}(p)$

We can observe, in Figure 3, that the function $\mathrm{M}(p)$ has a different behavior in four disjoint regions of the positive real line.

We should notice that $\mathrm{M}(p)$ is a function of products of gamma functions with arguments depending on $\frac{1}{p}$. Hence, the Stirling approximation is well-behaved for values close to the origin. Then, we have

$$\Gamma(x) = \sqrt{2\pi} x^{x-\frac{1}{2}} e^{-x} \left[1 + O\left(x^{-1}\right)\right], \quad x > 0, \tag{13}$$

see Gradshteyn & Ryzhik (1994), equation 8.327. The Stirling approximation is very accurate for large x, such as it is shown in equation (13).

Gurland (1956) showed that the gamma function satisfies the following inequality

$$\frac{\Gamma^2(\alpha+\delta)}{\Gamma(\alpha)\Gamma(\alpha+2\delta)} \leq \frac{\alpha}{\alpha+\delta^2}, \quad \alpha+\delta > 0, \quad \alpha > 0. \tag{14}$$

and by doing $\alpha = 1/p$ and $\delta = 1/p$ in equation (14) we can obtain the following inequality for $\mathrm{M}(p)$

$$\mathrm{M}(p) = \frac{\Gamma^2(2/p)}{\Gamma(1/p)\Gamma(3/p)} \leq \frac{p}{1+p}, \tag{15}$$

the equality is achieved when $p = 1$. This suggests that we can approximate the function $\mathrm{M}(p)$ by $\dfrac{b_1 p}{1 + b_2 p + b_3 p^2}$ around $p = 1$.

In summary, we have a good approximation close to $p = 0+$ by using Stirling asymptotic result, and by using Gurland's inequality we have another good approximation close to $p = 1$. However, there exist a region belonging to the segment $(0, 1)$ where we can find a better approximation for $\mathrm{M}(p)$ by using the following polynomial function $a_1 p^2 + a_2 p + a_3$. On the other hand, for values of $p > 1$ we have that the function $\mathrm{M}(p)$ has as an asymptote the horizontal line $\frac{3}{4}$, and we propose the following approximation $\frac{3}{4} - c_1 e^{-c_2 p + c_3 p^2}$ in this case.

We should observe that the four proposed functions can be easily inverted. The particular values for each of the constants a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 depend on each particular region and the specified accuracy for the approximation. For instance, for the Stirling approximation we specify an error smaller than 0.001.

Then, by using equation (13) and by taking $\Gamma(x) \cong \Gamma_\circ(x) = \sqrt{2\pi} x^{x-\frac{1}{2}} e^{-x}$, we have

$$\mathrm{M}(p) = \frac{\Gamma^2\left(\frac{2}{p}\right)}{\Gamma\left(\frac{1}{p}\right)\Gamma\left(\frac{3}{p}\right)} \cong \frac{\Gamma_\circ^2\left(\frac{2}{p}\right)}{\Gamma_\circ\left(\frac{1}{p}\right)\Gamma_\circ\left(\frac{3}{p}\right)} = \frac{1}{4} 3^{\frac{1}{2}\frac{p-6}{p}} 2^{\frac{4+p}{p}},$$

if additionally, we impose that

$$\left| \frac{1}{4} 3^{\frac{1}{2}\frac{p-6}{p}} 2^{\frac{4+p}{p}} - \mathrm{M}(p) \right| \leq 0.001,$$

then we should have $0 \leq p < 0.2771981$. This approximation becomes exact when $p = 0$. Similarly, by applying the least-squares goodness-of-fit, we have found that the approximation according to Gurland's inequality is suitable in the range $0.828012 \leq p < 2.631718$. In the same manner, the polynomial function approximation is adequate for the range $[0.\,277198\,1, 0.828012)$, and the exponential asymptotic approximation for $p \geq 2.631718$. Then, the function $\mathrm{M}(p)$ may be approximated as follows

$$\mathrm{M}^*(p) = \begin{cases} 3^{\frac{1}{2}\frac{p-6}{p}} 2^{\frac{4-p}{p}} & \text{if} \quad p \in [0, 0.277198\,1) \\ a_1 p^2 + a_2 p + a_3 & \text{if} \quad p \in [0.\,277198\,1, 0.828012) \\ \dfrac{b_1 p}{1 + b_2 p + b_3 p^2} & \text{if} \quad p \in [0.828012, 2.631\,718) \\ \frac{3}{4} - c_1 e^{-c_2 p + c_3 p^2} & \text{if} \quad p \in [2.631\,718, \infty), \end{cases}$$

where $a_1 = -0.535707356$, $a_2 = 1.168939911$, $a_3 = -0.1516189217$, $b_1 = 0.969442\,9$, $b_2 = 0.872753\,4$, $b_3 = 0.07350\,824$, $c_1 = 0.365515\,7$, $c_2 = 0.672353\,2$, $c_3 = 0.0338\,34$.

There is an excellent matching between the approximated function $\mathrm{M}^*(p)$ and the exact function $\mathrm{M}(p)$, as it is shown in Figure 4.

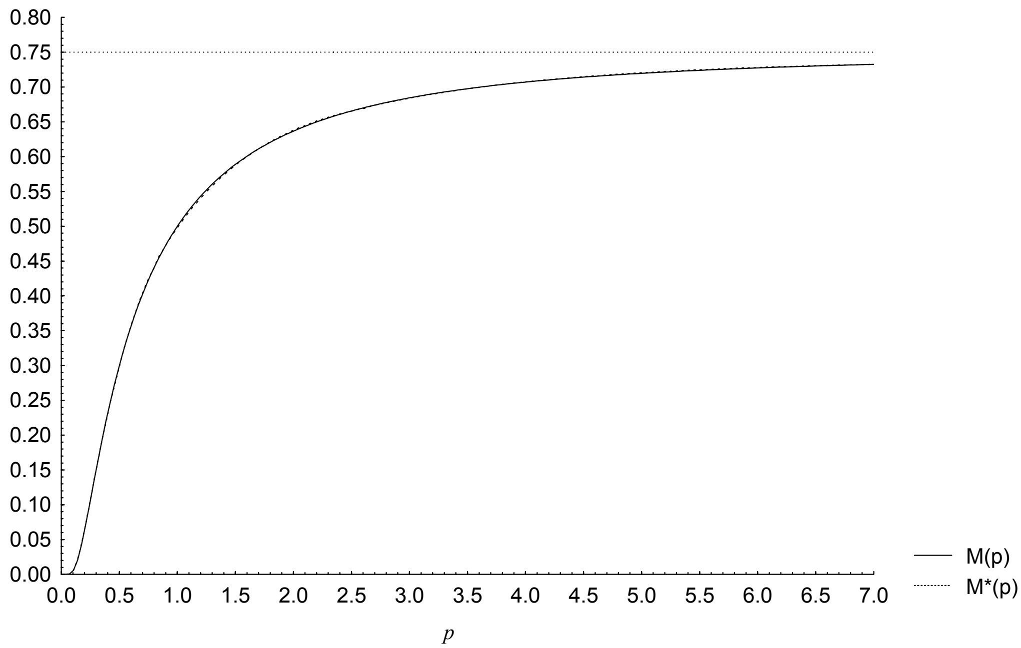

Figure 4: Solid line: $\mathrm{M}(p)$, dash line: $\mathrm{M}^*(p)$

The corresponding inverse function for $\mathrm{M}^*(p)$ is given by

$$p^*(k) = \begin{cases} 2\dfrac{\ln \frac{27}{16}}{\ln \frac{3}{4k^2}} & \text{if} \quad k \in (0, 0.131246) \\ \dfrac{1}{2a_1}\left(-a_2 + \sqrt{a_2^2 - 4a_1a_3 + 4a_1k}\right) & \text{if} \quad k \in [0.131246, 0.448994) \\ \dfrac{1}{2b_3k}\left(b_1 - b_2k - \sqrt{(b_1 - b_2k)^2 - 4b_3k^2}\right) & \text{if} \quad k \in [0.448994, 0.671256) \\ \dfrac{1}{2c_3}\left(c_2 - \sqrt{c_2^2 + 4c_3 \ln\left(\frac{3-4k}{4c_1}\right)}\right) & \text{if} \quad k \in \left[0.671256, \frac{3}{4}\right), \end{cases}$$

from which it is possible to find an approximated MME for p.

4 Confidence intervals

Since we do not have a sufficient statistic for p, it is not an easy task to build confidence intervals for this shape parameter. There are several alternatives to build approximated confidence intervals, namely the likelihood function for the interval, the Rao score statistic, and the Wald statistic, which are equivalent statistics on the order $O(n^{-1})$, however the confidence intervals built with these statistics have a poor performance since their covering probabilities depend strongly on the parameter p. Hence, we propose a numeric-analytic procedure to build such confidence intervals for p. Our method is based on the statistic sampled generalized Gaussian ratio function

$$\bar{\mathrm{M}}(x) = \frac{\left(\frac{1}{n}\sum_{i=1}^{n}|x_i|\right)^2}{\frac{1}{n}\sum_{i=1}^{n}x_i^2}$$

whose distribution does not depend on σ.

If we let n be fixed, then the *pdf* of $\bar{\mathrm{M}}(X)$ depends only on the parameter p. Now, by applying the *integral transformation theorem*, (Kalbfleisch (1985) pp. 215, Mood, *et al.*, pp. 387) we have that the new random variable

$$T = \mathrm{F}_{\bar{\mathrm{M}}}\left(\bar{\mathrm{M}}; p\right),$$

will be uniformly distributed on $(0,1)$. Therefore, in order to build a confidence interval for p we have to find a value of p such that

$$\Pr\left(\alpha_1 \leq \mathrm{F}_{\bar{\mathrm{M}}}\left(\bar{\mathrm{M}}; p\right) \leq \alpha_2\right) = 1 - \alpha,$$

where the most common values of α_1 and α_2 are $\alpha_1 = 1 - \alpha_2$, and $\frac{1}{2} < \alpha_2 < 1$.

We obtain the confidence intervals for p according to the Monte Carlo method for a specified confidence level and sample size. We have followed this procedure since it is quite involved to obtain a closed-form expression for the *pdf* of $\bar{\mathrm{M}}$, even for $n = 2$, as it can be seen in D.1. The distribution function

$$\mathrm{F}_{\bar{\mathrm{M}}}(m; p) = \Pr\left(\bar{\mathrm{M}} \leq m; p\right)$$

is evaluated by simulation.

We need to solve the equation $\Pr(\alpha/2 \leq \mathrm{F}_{\bar{\mathrm{M}}}(M; p) \leq 1 - \alpha/2)$ for p, and for an observed value M_o of $\bar{\mathrm{M}}$, then we proceed according to the following algorithm:

Assume that we have the observed value M_o, of $\bar{\mathrm{M}}$.

1. Fix an initial value $p = p_0$.

2. Obtain m samples of size n, $(x_{1,1}, x_{1,2}, ..., x_{1,n})$, $(x_{2,1}, x_{2,2}, ..., x_{2,n})$, ..., $(x_{m,1}, x_{m,2}, ..., x_{m,n})$, of absolute values of the GG *pdf* with $\mu = 0$, $p = p_0$ and $\sigma = 1$ (in fact the value $\sigma = 1$ is not important since the *pdf* of M is independent of σ). In the Appendix A we indicate how to simulate GG (μ, σ, p).

3. Evaluate $M_1, M_2, ..., M_m$, where $M_i = \frac{\left(\frac{1}{n}\sum_{j=1}^{n} x_{j,i}\right)^2}{\frac{1}{n}\sum_{j=1}^{n} x_{j,i}^2}$.

4. Evaluate the empirical distribution function of M, in M_o, by using

$$\begin{aligned}\bar{\mathrm{F}}_{\bar{\mathrm{M}}}(M_o) &= \frac{1}{m}(\#M_i \le M_o) \\ &= \frac{1}{m}\sum_{i=1}^{m} 1_{(-\infty, M_o]}(M_i)\end{aligned}$$

5. If $\bar{\mathrm{F}}_{\bar{\mathrm{M}}}(M_o) \approx \alpha/2$, then we select p_0 and the search is ended, otherwise we repeat step 1, with a different value for p_0.

In order to obtain the value p_1 such that $\bar{\mathrm{F}}_{\bar{\mathrm{M}}}(M_o) = 1 - \alpha/2$, we should repeat the previous procedure, and now in 5, we should ask for if $\bar{\mathrm{F}}_{\bar{\mathrm{M}}}(M_o) \approx 1 - \alpha/2$ is achieved.

The values p_0 and p_1 form the confidence interval of $100(1-\alpha)$ for p.

5 Application to the MP3 audio encoder.

There has been proposed many techniques to represent in digital form audio signals, with the purposes of storage and transmission as bits of information. The audio signals are essentially analog, however by making the transformation to a digital domain it is possible to minimize the number of bits and keeping, at the same time, an adequate quality level. Different applications require different tradeoffs depending on the specified quality levels. This fact has originated several encoding proposals for speech and audio signal. For instance, for speech applications in traditional telephone networks, in cellular networks, in toys, etc.. In recent years, the MP3 standard has became very popular for high-quality digital audio applications. This standard was proposed as a part of the MPEG-1 video encoder in 1992, and now is widely used in personal computers and transfer through Internet of music files. This compression scheme of audio signals is focused to achieve a similar quality as the uncompressed digital files of commercial compact discs, however achieving a high compression bit rate.

The MP3 standard audio encoder is composed of three layers with different complexity levels and bit rates. Each of the layers has basically the same structure, i.e. a filterbank of 32 polyphasic passband filters in a perfect reconstruction arrangement. The sampling frequency is 44.1 KHz for maximum quality, and it can process audio at 20KHz of bandwidth. In practical implementation of the MP3 encoder, there are some imperfections due to the actual filters cannot be implemented in an ideal manner. However, the maximum degradation does occur due to the quantizer stage, which restrict the amplitudes of the samples, at the output of each of the filters, to a finite set of values. In this scheme of analysis and reconstruction according to a filterbank decomposition, the samples at the output of each of the polyphasic filters are subsampled or decimated by a factor 32, then they are transformed by using the modified discrete cosine transform (MDCT), and the quantization stage is applied on the MDCT coefficients. In order to optimize these quantizers, it is necessary to know the *pdf* of the information sources, or equivalently the data at the output of the filterbank. In this work we will apply our statistical analysis to 28,657 samples obtained at the output of each of the 32 filters. It corresponds to 20.79 seconds of the musical piece *Carmina Burana.*

Table 1. $\bar{M}(x)$ represents the sampled values of the sggrf corresponding to the data of the musical piece *Carmina Burana*. Where $\bar{p}$ is the MME of p, which is obtained based on (12), whereas p_0 and p_1 are the extreme values of the 95% confidence interval of p. These values were obtained based on the algorithm of the previous section by taking $n = 28657$, $m = 500$, and $\sigma = 1$.

	$\bar{M}(x)$	$\bar{p}$	p_0	p_1	$p_1 - p_0$		$\bar{M}(x)$	$\bar{p}$	p_0	p_1	$p_1 - p_0$
0	0.5614	1.3211	1.2754	1.3415	0.0661	16	0.1196	0.2643	0.2454	0.2882	0.0428
1	0.5162	1.0798	1.0442	1.0966	0.0524	17	0.0919	0.2333	0.2124	0.2571	0.0447
2	0.4992	1.0044	0.9722	1.0208	0.0486	18	0.0750	0.2138	0.1949	0.2377	0.0428
3	0.5163	1.0806	1.0422	1.0927	0.0505	19	0.0622	0.1987	0.1755	0.2299	0.0544
4	0.4990	1.0037	0.9722	1.0220	0.0498	20	0.0581	0.1937	0.1716	0.2143	0.0427
5	0.4811	0.9311	0.9062	0.9509	0.0447	21	0.0583	0.1940	0.1716	0.2221	0.0505
6	0.4087	0.7109	0.6963	0.7352	0.0389	22	0.0650	0.2021	0.1794	0.2299	0.0505
7	0.3498	0.5867	0.5719	0.6069	0.0350	23	0.0609	0.1971	0.1755	0.2221	0.0466
8	0.3717	0.6290	0.61467	0.6496	0.0349	24	0.0667	0.2041	0.1813	0.2299	0.0486
9	0.3265	0.5452	0.5292	0.5622	0.0330	25	0.0705	0.2086	0.1871	0.2338	0.0467
10	0.2905	0.4869	0.4689	0.5020	0.0331	26	0.0716	0.2099	0.1871	0.2338	0.0467
11	0.2482	0.4248	0.4048	0.4437	0.0389	27	0.0670	0.2045	0.1832	0.2377	0.0545
12	0.2300	0.3996	0.3815	0.4164	0.0349	28	0.0529	0.1872	0.1638	0.2221	0.0583
13	0.2210	0.3876	0.3698	0.4048	0.0350	29	0.0390	0.1688	0.1424	0.1910	0.0486
14	0.1690	0.3217	0.3037	0.3426	0.0389	30	0.0490	0.1822	0.1599	0.2104	0.0505
15	0.1722	0.3256	0.3076	0.3465	0.0389	31	0.0531	0.1874	0.1638	0.2143	0.0505

In Figure 5, we can verify that our assumption of the mean value equal to zero is a reasonable figure for this application.

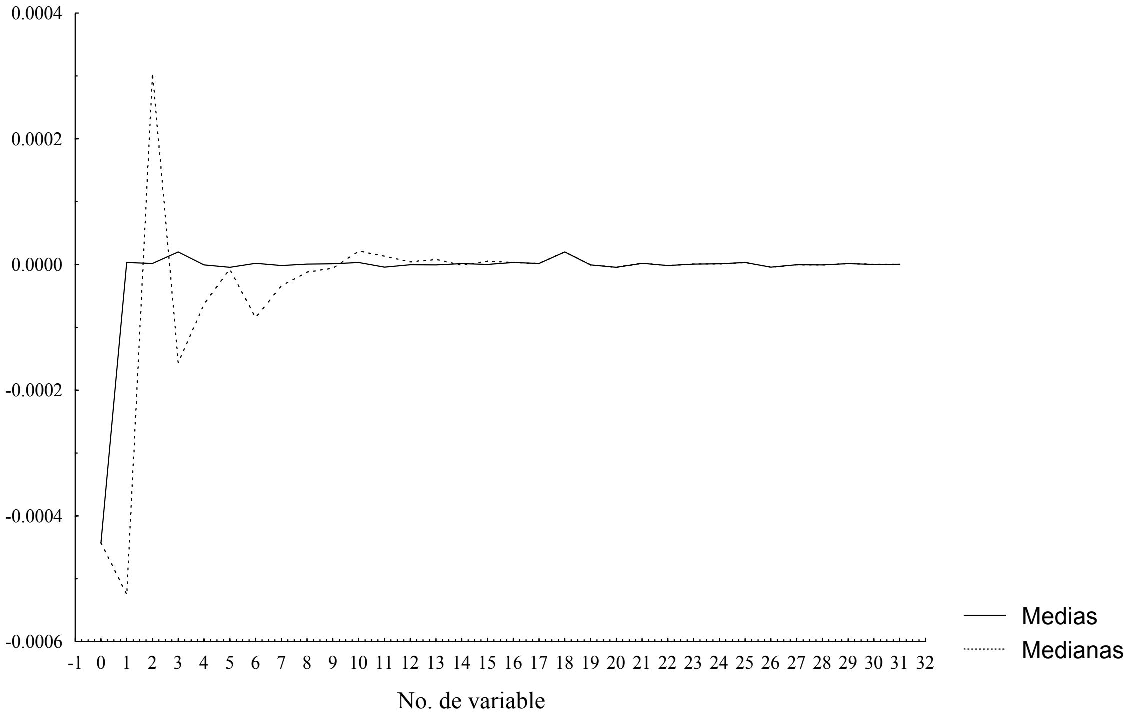

Figure 5. Means and medians corresponding to the data of each of the 32 bands.

In Tables 1 and 2 we can notice that the estimated p and σ values according to our method (12), result very accurate. This fact can be also verified through the confidence intervals shown in Figure 6.

Table 2. Standard deviation estimators (σ), for data corresponding to the 32 subbands, where $\hat{\sigma}$ denotes MLE of σ, obtained in (B.2), and $\bar{\sigma}$ denotes the MME of σ, obtained according to (11).

Col	$\hat{\sigma}$	$\bar{\sigma}$	Col	$\hat{\sigma}$	$\bar{\sigma}$
0	0.1522	0.1522	8	0.0045	0.0045
1	0.1156	0.1157	9	0.0043	0.0042
2	0.0624	0.0626	10	0.0039	0.0037
3	0.0339	0.0340	11	0.0037	0.0034
4	0.0222	0.0222	12	0.0031	0.0028
5	0.0133	0.0134	13	0.0028	0.0025
6	0.0084	0.0083	14	0.0026	0.0022
7	0.0061	0.0060	15	0.0022	0.0018
Col	$\hat{\sigma}$	$\bar{\sigma}$	Col	$\hat{\sigma}$	$\bar{\sigma}$
16	0.0025	0.0018	24	0.0014	0.0007
17	0.0029	0.0019	25	0.0012	0.0006
18	0.0031	0.0018	26	0.0010	0.0005
19	0.0032	0.0017	27	0.0009	0.0004
20	0.0028	0.0014	28	0.0010	0.0004
21	0.0025	0.0013	29	0.0012	0.0004
22	0.0023	0.0010	30	0.0007	0.0003
23	0.0018	0.0008	31	0.0005	0.0002

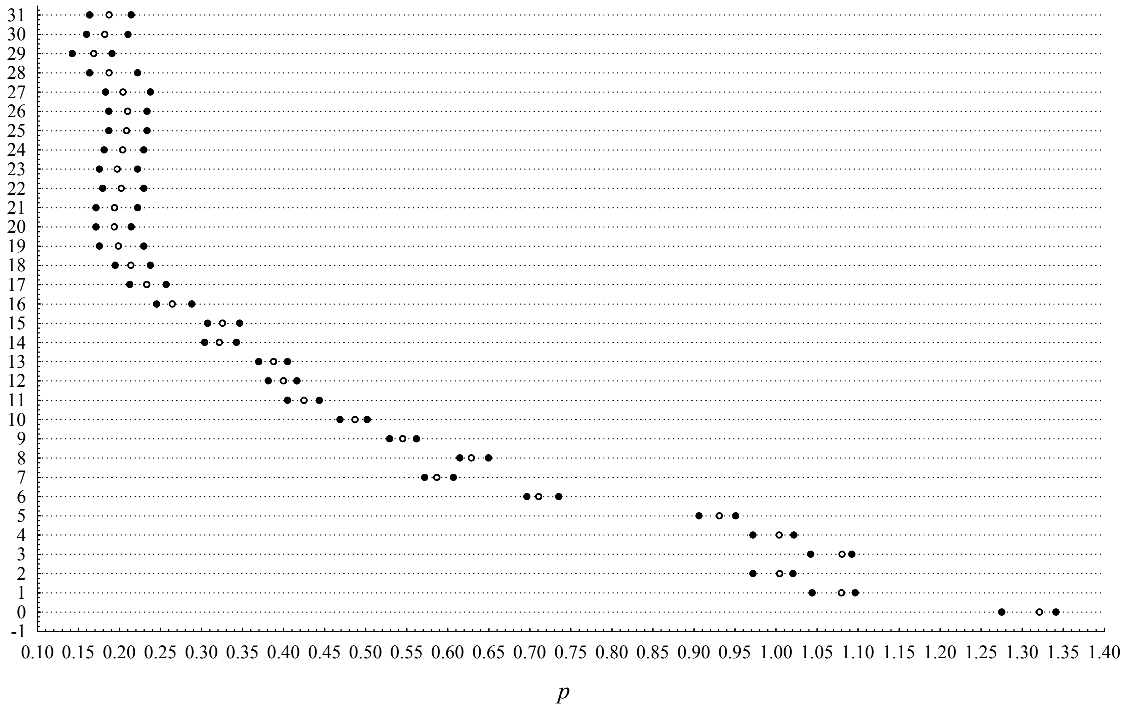

Figure 6. 95 % confidence interval for p. The black circles represents the extremes of the CI, and the white circles represents the MME of p.

6 Conclusion

It is important to point that the MLE of σ depends on p. We can observe, in our data analysis that the MME for σ decreases as p decreases (higher subband number), and in some cases the MLE is more than twice the value of the MME.

The covering probability of the CI obtained by the ML method depends on p, and we suggest to calculate minimum covering probability in this case. On the other hand, the calculation of the minimum covering probability for the generalized Gaussian model results very involved from the computational point of view, this is so since it is complicated to calculate the MLE for p. We recommend to use equation (12) and the algorithm proposed in section 4 in order to obtain simple point estimators and their associated CI.

As we mention above, if $\bar{\mathrm{M}}(X) \geq \frac{3}{4}$, then it is not possible to estimate p. This fact may suggest two possible scenarios: first, the actual value of p is large, and second, we are trying to fit the wrong distribution to the data. If we can obtain the MLE $\hat{p}$, and we obtain a very small value then we should not rely on the GG as the proper distribution, and we may get a better fit by using a different distribution, such as the generalized gamma, see Appendix E.

Acknowledgements

We would like to thank Ing. José I. Gómez Quiñones for providing the data analyzed in this work.

7 Appendix

A Some properties of the generalized Gaussian distribution

In this section we obtain some basic properties of the generalized Gaussian distribution.

The mean value of the GG distribution, μ, can be obtained as follows

$$\begin{aligned}
EX &= \frac{1}{2\Gamma(1+1/p)\,\mathrm{A}(p,\sigma)} \int_{-\infty}^{\infty} x e^{-\left|\frac{x-\mu}{\mathrm{A}(p,\sigma)}\right|^p} dx \\
&= \mu + \frac{1}{2\Gamma(1+1/p)\,\mathrm{A}(p,\sigma)} \int_{-\infty}^{\infty} (x-\mu)\, e^{-\left|\frac{x-\mu}{\mathrm{A}(p,\sigma)}\right|^p} dx \\
&= \mu + \frac{1}{2\Gamma(1+1/p)\,\mathrm{A}(p,\sigma)} \int_{-\infty}^{\infty} y e^{-\left|\frac{y}{\mathrm{A}(p,\sigma)}\right|^p} dy \\
&= \mu.
\end{aligned}$$

Now, let us make $\mu = 0$, and let $Y = |X|$, then the *pdf* of Y is given by

$$f_Y(y) = \frac{1}{\Gamma(1+1/p)\,\mathrm{A}(p,\sigma)} e^{-\frac{y^p}{[\mathrm{A}(p,\sigma)]^p}},$$

hence the absolute moments of X are

$$E|X|^r = EY^r = \frac{1}{\Gamma(1+1/p)\,\mathrm{A}(p,\sigma)} \int_0^{\infty} y^r e^{-\frac{y^p}{[\mathrm{A}(p,\sigma)]^p}} dy,\ r > 0.$$

We define the variable $w = \frac{y^p}{[\mathrm{A}(p,\sigma)]^p}$ then we have

$$
\begin{aligned}
EY^r &= \frac{1}{\Gamma(1+1/p)\,\mathrm{A}(p,\sigma)} \int_0^\infty [\mathrm{A}(p,\sigma)]^r\, w^{r/p} e^{-w} \mathrm{A}(p,\sigma)\, \frac{1}{p} w^{1/p-1} dw \\
&= \frac{[\mathrm{A}(p,\sigma)]^r}{p\Gamma(1+1/p)} \int_0^\infty w^{\frac{r+1}{p}-1} e^{-w} dw \\
&= \frac{[\mathrm{A}(p,\sigma)]^r}{p\Gamma(1+1/p)} \Gamma\left(\frac{r+1}{p}\right) \\
&= \left[\frac{\sigma^2\Gamma(1/p)}{\Gamma(3/p)}\right]^{r/2} \frac{\Gamma\left(\frac{r+1}{p}\right)}{\Gamma(1/p)} \qquad \text{(A.16)}
\end{aligned}
$$

In particular, the variance of X is given by

$$\mathrm{Var}(X) = E(X - EX)^2 = E(X-\mu)^2 = EY^2 = \sigma^2.$$

A.1 Generalized Gaussian distribution for $p = 0+$ and $p \to +\infty$

We have the following results

$$\lim_{p\to\infty} \Gamma(1+1/p) = 1,$$

and

$$\frac{\Gamma(1/p)}{\Gamma(3/p)} \searrow 3,\ p \to \infty,\ p \geq 9.1147,$$

the limit above can be obtained from (E.5).

When $-\sqrt{3}\sigma < x - \mu < \sqrt{3}\sigma$ it is achieved that

$$-1 < \frac{x-\mu}{A(p,\sigma)} < 1,$$

therefore, we obtain

$$\lim_{p\to\infty} \frac{|x-\mu|^p}{[A(p,\sigma)]^p} = \begin{cases} 0, & \text{if} \quad \mu - \sqrt{3}\sigma < x < \mu + \sqrt{3}\sigma, \\ +\infty, & \text{otherwise} \end{cases}$$

From the previous results, and by making $p \to +\infty$ we can find that the distribution of X is $U\left(\mu - \sqrt{3}\sigma, \mu + \sqrt{3}\sigma\right)$, *i.e.*,

$$\lim_{p \to +\infty} gg\left(x, \sigma, p\right) = \begin{cases} \frac{1}{2\sqrt{3}\sigma}, & \text{if} \quad \mu - \sqrt{3}\sigma < x < \mu + \sqrt{3}\sigma, \\ 0, & \text{otherwise} \end{cases}$$

it means, the generalized Gaussian *pdf* when $p = +\infty$ is

$$F_{GG}\left(x; \mu, \sigma, +\infty\right) = \begin{cases} 0, & \text{if} \quad x \leq \mu - \sqrt{3}\sigma \\ \frac{1}{2} + \frac{1}{2\sqrt{3}}\left(\dfrac{x-\mu}{\sigma}\right), & \text{if} \quad \mu - \sqrt{3}\sigma < x < \mu + \sqrt{3}\sigma, \\ 1, & \text{if} \quad x \geq \mu + \sqrt{3}\sigma. \end{cases}$$

We should observe that when $\mu = 0$, we have in the limit $E\left|X\right| = \frac{\sqrt{3}}{2}\sigma$ and $EX^2 = \sigma^2$, which implies that $\frac{(E|X|)^2}{EX^2} = \frac{3}{4}$.

When p approaches to zero for the right we have that

$$\lim_{p \to 0+} gg\left(x; \mu, \sigma, p\right) = \begin{cases} 0, & \text{if } x \neq \mu, \\ +\infty, & \text{if} \quad x = \mu. \end{cases}$$

From the previous limit it is easy to see that the generalized Gaussian *pdf* in $p = 0+$ is given by

$$F_{GG}\left(x; \mu, \sigma, 0+\right) = \begin{cases} 0, & \text{if } x < \mu, \\ 1, & \text{if} \quad x \geq \mu, \end{cases}$$

it means, when $p \to 0+$ the random variable $\mathrm{G}\left(\mu, \sigma, p\right)$ converges to a random variable with degenerate distribution in $x = \mu$.

A.2 Simulation of generalized Gaussian random variables

Let $X \sim \mathrm{GG}\left(\mu, \sigma, p\right)$. Now, let consider $\mu = 0$ and $Y = \left|X\right|$, then the *pdf* of Y is

$$f_Y\left(y; \sigma, p\right) = \frac{1}{\Gamma\left(1 + 1/p\right) \mathrm{A}\left(p, \sigma\right)} e^{-\frac{y^p}{[\mathrm{A}(p,\sigma)]^p}}. \tag{A.17}$$

Let Z be a gamma distributed random variable with *pdf*

$$g(z;\alpha,\lambda)=\frac{\alpha^{\lambda}}{\Gamma(\lambda)}z^{\lambda-1}e^{-\alpha z}, \tag{A.18}$$

it means, Z is gamma distributed with parameters α and λ, or equivalently $Z\sim \mathrm{G}(\alpha,\lambda)$.

Let $Z\sim \mathrm{G}(\alpha,\lambda)$, with $\alpha=[\mathrm{A}(p,\sigma)]^{-p}$, $\lambda=p^{-1}$. Then

$$\begin{aligned} f_Z(z) &= \frac{\left\{[\mathrm{A}(p,\sigma)]^{-p}\right\}^{1/p}}{\Gamma\left(\frac{1}{p}\right)}z^{\frac{1}{p}-1}e^{-[\mathrm{A}(p,\sigma)]^{-p}z} \\ &= \frac{1}{\Gamma\left(\frac{1}{p}\right)\mathrm{A}(p,\sigma)}z^{\frac{1}{p}-1}e^{-[\mathrm{A}(p,\sigma)]^{-p}z}. \end{aligned}$$

By letting $Y=Z^{1/p}$, we have $z=y^p$ and $dz=py^{p-1}$. Then the *pdf* of Y is given by

$$\begin{aligned} f_Y(y) &= \frac{1}{\Gamma\left(\frac{1}{p}\right)\mathrm{A}(p,\sigma)}\left(y^p\right)^{\frac{1}{p}-1}e^{-[\mathrm{A}(p,\sigma)]^{-p}y^p}py^{p-1} \\ &= \frac{1}{\Gamma\left(1+\frac{1}{p}\right)\mathrm{A}(p,\sigma)}e^{-[\mathrm{A}(p,\sigma)]^{-p}y^p}, \end{aligned}$$

therefore, the random variable Y has a *pdf* as is distributed as in (A.17).

Then, to simulate absolute values of a generalized Gaussian *pdf* with parameters σ and p, we should first simulate random variables $Z_i\sim \mathrm{G}\left(A^{-p},p^{-1}\right),\ i=1,...,n$ and based on these distributions we obtain the new variables $Y_i=Z^{1/p}$ which are distributed according to (A.17).

Obtain random variables with *pdf* (1) according to the method suggested by Michael, Schucany and Haas(1976):

1.- Simulate W from a random variable with a *pdf* of the absolute value of GG with $\mu=0$

2.- Make $Y=(-1)^b W$, where b is a Bernoulli random variable with parameter $\left(\frac{1}{2}\right)$

3.- Define $X=\mu+Y$, $\mu\in\mathbb{R}$.

Then the random variable X has a *pdf* as (1).

B Maximum likelihood

The maximum likelihood function of μ, σ and p is given by

$$L\left(p,\sigma;X\right)=\left[\Gamma\left(1+1/p\right)\mathrm{A}\left(p,\sigma\right)\right]^{-n}\exp\left\{-\left[\mathrm{A}\left(p,\sigma\right)\right]^{-p}\sum_{i=1}^{n}\left|x_i-\mu\right|^p\right\},$$

and its corresponding log-likelihood function is given by

$$\ell\left(p,\mu,\sigma;X\right)=-n\ln\left[\Gamma\left(1+1/p\right)\mathrm{A}\left(p,\sigma\right)\right]-\frac{1}{\left[\mathrm{A}\left(p,\sigma\right)\right]^p}\sum_{i=1}^{n}\left|x_i-\mu\right|^p.$$

If we make $Y=|X|$ and $\mu=0$ then the log-likelihood function of σ and p is given by

$$\ell\left(p,\sigma;X\right)=-n\ln\left[\Gamma\left(1+1/p\right)\mathrm{A}\left(p,\sigma\right)\right]-\frac{1}{\left[\mathrm{A}\left(p,\sigma\right)\right]^p}\sum_{i=1}^{n}x_i^p. \tag{B.1}$$

The MLE of σ can be obtained by solving the following equation for σ,

$$\frac{d}{d\sigma}\ell\left(p,\sigma;X\right)=-\frac{n}{\sigma}+\frac{p}{\sigma^{p+1}}\left[\frac{\Gamma\left(1/p\right)}{\Gamma\left(3/p\right)}\right]^{-p/2}\sum_{i=1}^{n}x_i^p=0,$$

hence, we have

$$\hat{\sigma}=\left[\frac{\Gamma\left(3/p\right)}{\Gamma\left(1/p\right)}\right]^{1/2}\left(\frac{p}{n}\sum_{i=1}^{n}x_i^p\right)^{1/p}. \tag{B.2}$$

C Inequalities for ratios of sums

In this section we show that the range of $\bar{\mathrm{M}}\left(x\right)$ is $\left(\frac{1}{n},1\right)$, and we show that

$$1\leq\frac{\frac{1}{n}\sum\limits_{i=1}^{n}\left(x_i-\bar{x}\right)^r}{\left[\frac{1}{n}\sum\limits_{i=1}^{n}\left(x_i-\bar{x}\right)^2\right]^{r/2}}\leq n^{\frac{r}{2}-1}$$

The bound with value 1 can be obtained from the Hölder inequality, and the other one by using algebra of series.

The Hölder inequality can be described as follows:

Let $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ be two sets of nonnegative real numbers, and assume $\frac{1}{p} + \frac{1}{q} = 1$, with $p > 1$, then

$$\left(\sum_{i=1}^{n} a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} b_i^q\right)^{1/q} \geq \sum_{i=1}^{n} a_i b_i. \tag{C.1}$$

The equality is satisfied if and only if the successions $a_1^p, a_2^p, ..., a_n^p$ and $b_1^q, b_2^q, ..., b_n^q$ are proportional to each other.

C.1 Range of $\bar{\mathrm{M}}(x)$

Let $x_1, x_2, ..., x_n$ be positive random variables and assume $b_i = \frac{1}{n}$, $i = 1, 2, ..., n$. Then by using Hölder inequality, with $p = q = 2$, we have

$$\left(\sum_{i=1}^{n} x_i^2\right)^{1/2} \left(\sum_{i=1}^{n} \frac{1}{n^2}\right)^{1/2} \geq \sum_{i=1}^{n} x_i \frac{1}{n},$$

from the previous inequality we obtain

$$\left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right)^{1/2} \geq \frac{1}{n}\sum_{i=1}^{n} x_i \geq 0$$

therefore

$$0 \leq \frac{\left(\frac{1}{n}\sum\limits_{i=1}^{n} x_i\right)^2}{\frac{1}{n}\sum\limits_{i=1}^{n} x_i^2} \leq 1. \tag{C.2}$$

It means

$$0 \leq \bar{\mathrm{M}}(x) \leq 1.$$

Now, we will show, if $x_1 > 0, ..., x_n > 0$, then the range of $\bar{\mathrm{M}}(x)$ is given by

$$\frac{1}{n} \leq \bar{\mathrm{M}}(x) \leq 1.$$

This is a straightforward result that can be obtained by observing that

$$\left(\sum_{i=1}^{n} x_i\right)^2 = \sum_{i=1}^{n} x_i^2 + \text{ something positive},$$

then

$$\left(\sum_{i=1}^{n} x_i\right)^2 \geq \sum_{i=1}^{n} x_i^2,$$

or, equivalently

$$\frac{\left(\sum\limits_{i=1}^{n} x_i\right)^2}{n\sum\limits_{i=1}^{n} x_i^2} \geq \frac{1}{n},$$

therefore, from the result (C.2), we can obtain

$$\frac{1}{n} \leq \frac{\left(\sum\limits_{i=1}^{n} x_i\right)^2}{n\sum\limits_{i=1}^{n} x_i^2} \leq 1.$$

In summary, the range of $\bar{\mathrm{M}}\left(x\right)$ is given by $\left[\frac{1}{n}, 1\right]$.

C.2 Range of ratio of sums

Let us consider $r \geq 3$ and suppose that $y_i = \left|x_i - \bar{x}\right|$, therefore $y_i \geq 0$. Then, by applying Hölder inequality (C.1) we have

$$\left(\sum_{i=1}^{n}\left(\frac{1}{n}y_i^2\right)^{\frac{r}{2}}\right)^{\frac{2}{r}}\left(\sum_{i=1}^{n} 1\right)^{1-\frac{2}{r}} \geq \sum_{i=1}^{n}\frac{1}{n}y_i^2,$$

then we can obtain the following inequality

$$\left(\frac{1}{n}\sum_{i=1}^{n} y_i^r\right)^{\frac{2}{r}} \geq \frac{1}{n}\sum_{i=1}^{n} y_i^2$$

which implies

$$\frac{\frac{1}{n}\sum_{i=1}^{n} y_i^r}{\left(\frac{1}{n}\sum_{i=1}^{n} y_i^2\right)^{r/2}} \geq 1. \tag{C.3}$$

Now, assume r even, then the following equation is satisfied

$$\left(\frac{1}{n}\sum_{i=1}^{n} y_i^2\right)^{r/2} = \frac{1}{n^{r/2}}\sum_{i=1}^{n} y_i^r + \text{something positive},$$

hence

$$\left(\frac{1}{n}\sum_{i=1}^{n} y_i^2\right)^{r/2} \geq \frac{1}{n^{r/2}}\sum_{i=1}^{n} y_i^r,$$

therefore

$$\frac{\frac{1}{r}\sum_{i=1}^{n} y_i^r}{\left(\frac{1}{n}\sum_{i=1}^{n} y_i^2\right)^{r/2}} \leq n^{\frac{r}{2}-1}. \tag{C.4}$$

And from the inequalities (C.3) and (C.4) we can obtain

$$1 \leq \frac{\frac{1}{r}\sum_{i=1}^{n} y_i^r}{\left(\frac{1}{n}\sum_{i=1}^{n} y_i^2\right)^{r/2}} \leq n^{\frac{r}{2}-1}.$$

This inequality is valid for all $r \geq 2$. When $r = 1$ we have a similar relationship, however valid only for this case. We know that

$$n^{-1/2} \leq \frac{\frac{1}{n}\sum_{i=1}^{n} X_i}{\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right)^{1/2}} \leq 1.$$

Or, by writing this equation in a similar manner as above

$$1 \leq \frac{\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right)^{1/2}}{\frac{1}{n}\sum_{i=1}^{n} X_i} \leq \sqrt{n}.$$

By substituting y_i by $|x_i - \bar{x}|$ we have

$$1 \leq \frac{\frac{1}{n}\sum_{i=1}^{n} |x_i - \bar{x}|^r}{\left[\frac{1}{n}\sum_{i=1}^{n} |x_i - \bar{x}|^2\right]^{r/2}} \leq n^{\frac{r}{2}-1}.$$

D *Pdf* of $\bar{\mathrm{M}}(X)$ for $n = 2$

Let X_1 and X_2 be independent and identically distributed random variables, with *pdf* $\mathrm{GG}(\sigma, p)$. Assume $Y_1 = |X_1|$ and $Y_2 = |X_2|$, then

$$\bar{\mathrm{M}}(X) = \frac{\frac{1}{4}(Y_1 + Y_2)^2}{\frac{1}{2}(Y_1^2 + Y_2^2)} = \frac{(Y_1 + Y_2)^2}{2(Y_1^2 + Y_2^2)}$$

Define the random variable $W = Y_1$ to obtain the *pdf* of $\bar{\mathrm{M}}$. The inverse function of the transformation $(Y_1, Y_2) \to \left(\bar{\mathrm{M}}, W\right)$ is

$$y_1 = w, \quad y_{2\pm} = \frac{1}{2m-1}\left(1 \pm 2\sqrt{m - m^2}\right) w$$

From the following partial derivatives

$$\frac{\partial y_1}{\partial m} = 0, \quad \frac{\partial y_1}{\partial w} = 1, \quad \frac{\partial y_{2\pm}}{\partial m} = \frac{\mp 1 - 2\sqrt{m - m^2}}{(2m-1)^2 \sqrt{m - m^2}} w,$$

$$\frac{\partial y_{2\pm}}{\partial w} = \frac{1}{2(2m-1)}\left(2 \pm 4\sqrt{m - m^2}\right),$$

we can obtain the jacobian of this transformation as

$$J_{\pm} = \begin{vmatrix} 0 & 1 \\ \frac{\partial y_{2\pm}}{\partial m} & \frac{\partial y_{2\pm}}{\partial w} \end{vmatrix} = -\frac{\partial y_{2\pm}}{\partial m}.$$

The joint *pdf* of Y_1 and Y_2 is given by

$$f_{Y_1, Y_2}(y_1, y_2) = \frac{1}{[\Gamma(1 + 1/p) A]^2} e^{-\frac{y_1^p}{A^p} - \frac{y_2^p}{A^p}},$$

then we obtain the joint *pdf* of M and W as

$$\begin{aligned} f(m, w) &= \frac{1}{[\Gamma(1 + 1/p) A]^2} \frac{1 + 2\sqrt{m - m^2}}{(2m-1)^2 \sqrt{m - m^2}} w \exp\left\{-\frac{w^p}{A^p}\left[1 + \frac{2 + 4\sqrt{m - m^2}}{2(2m-1)}\right]\right\} \\ &+ \frac{1}{[\Gamma(1 + 1/p) A]^2} \frac{1 - 2\sqrt{m - m^2}}{(2m-1)^2 \sqrt{m - m^2}} w \exp\left\{-\frac{w^p}{A^p}\left[1 + \frac{2 - 4\sqrt{m - m^2}}{2(2m-1)}\right]\right\}. \end{aligned}$$

In order to obtain the marginal *pdf* of M we use the change of variable

$$U_{\pm} = \frac{w}{A}\left[1 + \frac{2 \pm 4\sqrt{M - M^2}}{2(2M-1)}\right]^{1/p} = \frac{w}{A}\left[\frac{4M \pm 4\sqrt{M - M^2}}{2(2M-1)}\right]^{1/p}.$$

Then

$$\begin{aligned} f(m,u) &= \frac{1}{[\Gamma(1+1/p)A]^2}\frac{1+2\sqrt{m-m^2}}{(2m-1)^2\sqrt{m-m^2}}A^2\left[\frac{4m+4\sqrt{m-m^2}}{2(2m-1)}\right]^{-2/p} ue^{-u^p} \\ &\quad + \frac{1}{[\Gamma(1+1/p)A]^2}\frac{1-2\sqrt{m-m^2}}{(2m-1)^2\sqrt{m-m^2}}A^2\left[\frac{4m-4\sqrt{m-m^2}}{2(2m-1)}\right]^{-2/p} ue^{-u^p}, \end{aligned}$$

we have $\int_0^\infty ze^{-z^p}dz = \frac{\Gamma\left(\frac{2}{p}\right)}{p}$ then we finally obtain that the *pdf* of M is given by

$$\begin{aligned} f_M(m) &= \frac{\Gamma\left(\frac{2}{p}\right)}{p[\Gamma(1+1/p)]^2}\frac{1+2\sqrt{m-m^2}}{(2m-1)^2\sqrt{m-m^2}}\left[\frac{4m+4\sqrt{m-m^2}}{2(2m-1)}\right]^{-2/p} \\ &\quad + \frac{\Gamma\left(\frac{2}{p}\right)}{p[\Gamma(1+1/p)A]^2}\frac{1-2\sqrt{m-m^2}}{(2m-1)^2\sqrt{m-m^2}}A^2\left[\frac{4m-4\sqrt{m-m^2}}{2(2m-1)}\right]^{-2/p}, \end{aligned} \tag{D.1}$$

where $\frac{1}{2} \leq m \leq 1$.

E Alternatives to the generalized Gaussian distribution

E.1 Generalized gamma distribution

Let $X \sim \text{GammaG}(a,d,p)$, i.e.,

$$f_X(x;a,d,p) = \frac{p}{a\Gamma\left(\frac{d}{p}\right)}\left(\frac{x}{a}\right)^{d-1} e^{-\left(\frac{x}{a}\right)^p}, \tag{E.1}$$

thus, the r-th moment is given by

$$\begin{aligned} \mathrm{E}X^r &= \frac{p}{a^d\Gamma\left(\frac{d}{p}\right)}\int_0^\infty x^{r+d-1}e^{-\left(\frac{x}{a}\right)^p}dx \\ &= a^r\frac{\Gamma\left(\frac{r+d}{p}\right)}{\Gamma\left(\frac{d}{p}\right)}. \end{aligned}$$

Hence

$$\mathrm{M}(d,p)=\frac{(\mathrm{E}X)^2}{\mathrm{E}X^2}=\frac{\left(a\frac{\Gamma\left(\frac{1+d}{p}\right)}{\Gamma\left(\frac{d}{p}\right)}\right)^2}{a^2\frac{\Gamma\left(\frac{2+d}{p}\right)}{\Gamma\left(\frac{d}{p}\right)}}=\frac{\Gamma^2\left(\frac{1+d}{p}\right)}{\Gamma\left(\frac{d}{p}\right)\Gamma\left(\frac{2+d}{p}\right)}, \tag{E.2}$$

and

$$\lim_{p\to\infty}\mathrm{M}(d,p)=\frac{d\,(2+d)}{(d+1)^2},\quad \lim_{d\to\infty}\lim_{p\to\infty}\mathrm{M}(d,p)=1.$$

The *pdf* for the absolute value of the generalized Gaussian is given by

$$f(x,\sigma,p)=\frac{1}{\Gamma(1+1/p)\ \mathrm{A}(p,\sigma)}e^{-\frac{x^p}{[\mathrm{A}(p,\sigma)]^p}},$$

in this case $\mathrm{A}(p,\sigma)$ is a scale parameter, and it is given by $\mathrm{A}(p,\sigma)=\left[\frac{\sigma^2\Gamma(1/p)}{\Gamma(3/p)}\right]^{1/2}$.

Similarly to the generalized Gaussian distribution we can take a in (E.1) as

$$a=\mathrm{A}(d,p,\sigma)=\left[\frac{\sigma^2\Gamma\left(\frac{d}{p}\right)}{\Gamma\left(\frac{d+2}{p}\right)}\right]^{1/2},$$

which allow us to obtain $EX^2=\sigma^2$.

E.2 Limit values of $\mathrm{M}(d,p)$

In this section, we will show that $\lim_{p\to\infty}\mathrm{M}(d,p)=\frac{d(2+d)}{(d+1)^2}$ and $\lim_{p\to 0+}\mathrm{M}(d,p)=0$.

From Gradshteyn & Ryzhik, (1994), equation 8.321 we have

$$\Gamma(z+1)=\sum_{k=0}^{\infty}c_k z^k, \tag{E.3}$$

where

$$c_0=1, c_{n+1}=\frac{\sum_{k=0}^{n}(-1)^{k+1}s_{k+1}c_{n-k}}{n+1}; s_1=\gamma, s_n=\zeta(n) \text{ for } \quad n\geq 2, |z|<1$$

γ is the Euler's constant, $\gamma=\lim_{s\to\infty}\left(\sum_{m=1}^{s}\frac{1}{m}-\ln s\right)\simeq 0.5772156649$, and $\zeta(x)=\sum_{n=1}^{\infty}\frac{1}{n^x}$.

By using the equation (E.3) we have that an expansion of $\Gamma(x)$ around 0 is given by

$$\Gamma(x) = x^{-1} - \gamma + \left(\frac{1}{12}\pi^2 + \frac{1}{2}\gamma^2\right)x + \cdots . \tag{E.4}$$

From (E.4) we have that

$$\lim_{x\to 0^+}\frac{\Gamma(ax)}{\Gamma(bx)} = \lim_{x\to 0^+}\frac{\frac{1}{a} - x\gamma + b_1x^2 + b_2x^3 + \cdots}{\frac{1}{b} - x\gamma + b_1'x^2 + b_2'x^3 + \cdots} = \frac{\frac{1}{a}}{\frac{1}{b}} = \frac{b}{a}. \tag{E.5}$$

Limit when $p \to \infty$ Let $\mathrm{M}(d,p)$ the function defined in (E.2), then

$$\lim_{p\to\infty}\mathrm{M}(d,p) = \lim_{p\to 0+}\mathrm{M}\left(d,\frac{1}{p}\right).$$

Now,

$$\mathrm{M}\left(d,\frac{1}{p}\right) = \frac{\Gamma^2[(1+d)p]}{\Gamma(dp)\Gamma[(2+d)p]} = \frac{\Gamma[(1+d)p]}{\Gamma(dp)}\frac{\Gamma[(1+d)p]}{\Gamma[(2+d)p]},$$

by applying the result (E.5) we have

$$\begin{aligned}\lim_{p\to\infty}\mathrm{M}(d,p) &= \lim_{p\to 0+}\frac{\Gamma[(1+d)p]}{\Gamma(dp)}\lim_{p\to 0+}\frac{\Gamma[(1+d)p]}{\Gamma[(2+d)p]}\\ &= \frac{d}{(1+d)}\frac{(2+d)}{(1+d)}\\ &= \frac{d(2+d)}{(1+d)^2}.\end{aligned}$$

Limit of $M(d,p)$ when $p \to 0+$ **Case** $d = 1$. By using Stirling approximation (13) we have

$$\begin{aligned}\frac{\Gamma^2\left(\frac{2}{p}\right)}{\Gamma\left(\frac{1}{p}\right)\Gamma\left(\frac{3}{p}\right)} &= \frac{\left(\sqrt{2\pi}\left(\frac{2}{p}\right)^{\frac{2}{p}-\frac{1}{2}}e^{-\frac{2}{p}}\right)^2}{\left(\sqrt{2\pi}\left(\frac{1}{p}\right)^{\frac{1}{p}-\frac{1}{2}}e^{-\frac{1}{p}}\right)\left(\sqrt{2\pi}\left(\frac{3}{p}\right)^{\frac{3}{p}-\frac{1}{2}}e^{-\frac{3}{p}}\right)}[1+O(p)], p\to 0+\\ &= \frac{1}{2}16^{\frac{1}{p}}3^{\frac{1}{2}\frac{-6+p}{p}}[1+O(p)], \quad p\to 0+.\end{aligned}$$

Hence,

$$\begin{aligned}\lim_{p\to 0+} M(1,p) &= \lim_{p\to 0+} \frac{\Gamma^2\left(\frac{2}{p}\right)}{\Gamma\left(\frac{1}{p}\right)\Gamma\left(\frac{3}{p}\right)} \\ &= \lim_{p\to 0+} \frac{1}{2} 16^{\frac{1}{p}} 3^{\frac{1}{2}\frac{-6+p}{p}}.\end{aligned}$$

In order to calculate the previous limit, we should notice that

$$\frac{1}{2}16^{\frac{1}{p}}3^{-\frac{3}{p}+\frac{1}{2}} = \frac{1}{2}3^{\frac{1}{2}}\frac{16^{\frac{1}{p}}}{3^{\frac{3}{p}}} = \frac{1}{2}3^{\frac{1}{2}}\left(\frac{16^{1/3}}{3}\right)^{\frac{3}{p}},$$

and since $\frac{16^{1/3}}{3} \cong 0.839\,95 < 1$, we obtain

$$\begin{aligned}\lim_{p\to 0+}\left(\frac{16^{1/3}}{3}\right)^{\frac{3}{p}} &= \lim_{p\to\infty}\left(\frac{16^{1/3}}{3}\right)^{3p} \\ &= 0\end{aligned}$$

Case $d > 1$. We will use the Stirling approximation (13) again

$$\begin{aligned}\mathrm{M}\left(d,\frac{1}{p}\right) &= \frac{\Gamma^2\left[(1+d)p\right]}{\Gamma(dp)\Gamma\left[(2+d)p\right]} \\ &= \frac{\left\{\sqrt{2\pi}\left[(1+d)p\right]^{(1+d)p-\frac{1}{2}}e^{-(1+d)p}\right\}^2}{\sqrt{2\pi}(dp)^{dp-\frac{1}{2}}e^{-dp}\sqrt{2\pi}\left((2+d)p\right)^{(2+d)p-\frac{1}{2}}e^{-(2+d)p}}\left[1+O\left(p^{-1}\right)\right] \\ &= \frac{(1+2d+d^2)^{p+dp}}{(1+d)d^{dp-\frac{1}{2}}(4+4d+d^2)^p(2+d)^{dp-\frac{1}{2}}}\left[1+O\left(p^{-1}\right)\right]\end{aligned}$$

$$\begin{aligned}\lim_{p\to 0+}\mathrm{M}(d,p) &= \lim_{p\to\infty}\mathrm{M}\left(d,\frac{1}{p}\right) = \lim_{p\to\infty}\frac{\Gamma^2\left[(1+d)p\right]}{\Gamma(dp)\Gamma\left[(2+d)p\right]} \\ &= \lim_{p\to\infty}\frac{(1+2d+d^2)^{p+dp}}{(1+d)d^{dp-\frac{1}{2}}(4+4d+d^2)^p(2+d)^{dp-\frac{1}{2}}}\left[1+O\left(p^{-1}\right)\right] \\ &= \lim_{p\to\infty}\frac{d^{2p(1+d)}}{(1+d)d^{2dp-1}d^{2p}}\left[1+O\left(p^{-1}\right)\right] \\ &= \lim_{p\to\infty}\frac{d^{2p(1+d)}}{(1+d)d^{2p(d+1)-1}}\left[1+O\left(p^{-1}\right)\right] \\ &= \lim_{p\to\infty}\frac{d}{1+d}\left[1+O\left(p^{-1}\right)\right] \\ &= 0.\end{aligned}$$

F Tables

TABLE F.1: Values of $n^* = \min\left\{n : \Pr\left(\bar{M}\left(X\right) \geq \frac{3}{4}; p\right) > 0.05\right\}, p \in (0.2, 10), \bar{M}$ is defined in (10). The value of n^* is obtained through a search procedure and simulation. For each sample size of *iid* random variables $\mathrm{GG}\left(0,1,p\right)$ we did 10000 repetitions of the Monte Carlo experiment. Each random variable $\mathrm{GG}\left(0,1,p\right)$ was simulated according to the procedure described in the Appendix A, pp. 24.

TABLE F.2: Mean and median values for each of data output of the 32 filters of the musical piece *Carmina Burana*.

TABLE F.3: *25 observations of the* pdf $\mathrm{GG}\ (0,1,3)$. Simulated by using the procedure described on page 24.

TABLE F.4: *25 observations of the* pdf $\mathrm{GG}\ (0,1,1)$. Simulated values obtained through the procedure described at the end of Appendix A, (page 24).

p	n^*	p	n^*	p	n^*	p	n^*	p	n^*
0.2	5	2.2	31	4.2	138	6.2	389	8.2	1046
0.4	6	2.4	38	4.4	162	6.4	439	8.4	1071
0.6	8	2.6	42	4.6	179	6.6	479	8.6	1163
0.8	8	2.8	47	4.8	196	6.8	541	8.8	1273
1.0	11	3.0	61	5.0	216	7.0	620	9.0	1324
1.2	13	3.2	73	5.2	234	7.2	653	9.2	1482
1.4	15	3.4	77	5.4	272	7.4	676	9.4	1556
1.6	20	3.6	95	5.6	294	7.6	811	9.6	1690
1.8	23	3.8	100	5.8	347	7.8	877	9.8	1816
2.0	26	4.0	123	6.0	364	8.0	942	10.0	2019

Table F.1. Values of $n^* = \min\left\{n : \Pr\left(\bar{M}\left(X\right) \geq \frac{3}{4}; p\right) > 0.05\right\}, p \in (0.2, 10)$.

	Mean	Median		Mean	Median
0	-0.0^3443	-0.0^3443	16	0.0^5312	0.0^5312
1	0.0^5312	-0.0^3525	17	0.0^5169	0.0^5169
2	0.0^5169	0.0^3304	18	0.0^4199	0.0^4199
3	0.0^4199	-0.0^3156	19	-0.0^6619	-0.0^6619
4	-0.0^6619	-0.0^4627	20	-0.0^5445	-0.0^5445
5	-0.0^5445	-0.0^5736	21	0.0^5173	0.0^5173
6	0.0^5173	-0.0^4847	22	-0.0^5152	-0.0^5152
7	-0.0^5152	-0.0^4340	23	0.0^6693	0.0^6693
8	0.0^6693	-0.0^4120	24	0.0^5106	0.0^5106
9	0.0^5106	-0.0^5593	25	0.0^5323	0.0^5323
10	0.0^5323	0.0^4213	26	-0.0^5424	-0.0^5424
11	-0.0^5424	0.0^4133	27	-0.0^6334	-0.0^6334
12	-0.0^6334	0.0^5423	28	-0.0^6617	-0.0^6617
13	-0.0^6617	0.0^5823	29	0.0^5148	0.0^5148
14	0.0^5148	-0.0^5101	30	0.0^6270	0.0^6270
15	0.0^6270	0.0^5544	31	0.0^6507	0.0^6507

Table F.2. Mean and median values for each of the output data of the 32 filters of the MP3 audio encoder.

x_{1-5}	x_{6-10}	x_{11-15}	x_{16-20}	x_{21-25}
1.23095	−1.18200	1.45424	1.98366	0.80739
1.01731	1.13537	−0.82535	−0.86127	−0.67881
1.29247	1.11843	0.67120	1.43192	−0.15525
0.00224	0.15930	1.79635	−0.84292	−0.73295
−1.45834	0.78456	−0.82832	−0.59772	0.11567

Table F.3. *25 observations of the* pdf $\mathrm{GG}(0,1,3)$.

z_{1-5}	z_{6-10}	z_{11-15}	z_{16-20}	z_{21-25}
−0.43141	1.29497	−0.64378	0.90504	1.07424
0.99215	0.46702	−0.71231	0.39473	−0.43733
−0.14126	−0.60484	−0.21146	0.14831	0.53406
1.04025	1.00121	−0.62999	0.85679	−0.80782
−0.52153	−0.63275	−0.87137	−0.66678	−0.59262

Table F.4. *25 observations of the* pdf $\mathrm{GG}(0,1,1)$.

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