Characterization of Frequency Stability *

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Abstract—Consider a signal generator whose instantaneous output voltage $V(t)$ may be written as

$$V(t) = [V_0 + \epsilon(t)] \sin [2\pi\nu_0 t + \varphi(t)]$$

where V_0 and ν_0 are the nominal amplitude and frequency, respectively, of the output. Provided that $\epsilon(t)$ and $\dot{\varphi}(t) = (d\varphi/(dt)$ are sufficiently small for all time t, one may define the fractional instantaneous frequency deviation from nominal by the relation

$$y(t) \equiv \frac{\dot{\varphi}(t)}{2\pi\nu_0}.$$

A proposed definition for the measure of frequency stability is the spectral density $S_y(f)$ of the function $y(t)$ where the spectrum is considered to be one sided on a per hertz basis.

An alternative definition for the measure of stability is the infinite time average of the sample variance of two adjacent averages of $y(t)$; that is, if

$$\bar{y}_k = \frac{1}{\tau} \int_{t_k}^{t_k+\tau} y(t)\, dt$$

where τ is the averaging period, $t_{k+1} = t_k + T$, $k = 0, 1, 2 \cdots$, t_0 is arbitrary, and T is the time interval between the beginnings of two successive measurements of average frequency; then the second measure of stability is

$$\sigma_y^2(\tau) \equiv \left\langle \frac{(\bar{y}_{k+1} - \bar{y}_k)^2}{2} \right\rangle$$

where $\langle\ \rangle$ denotes infinite time average and where $T = \tau$.

In practice, data records are of finite length and the infinite time averages implied in the definitions are normally not available; thus estimates for the two measures must be used. Estimates of $S_y(f)$ would be obtained from suitable averages either in the time domain or the frequency domain. An obvious estimate for $\sigma_y^2(\tau)$ is

$$\sigma_y^2(\tau) \approx \frac{1}{m} \sum_{k=1}^{m} \frac{(\bar{y}_{k+1} - \bar{y}_k)^2}{2}.$$

Parameters of the measuring system and estimating procedure are of critical importance in the specification of frequency stability. In practice, one should experimentally establish confidence limits for an estimate of frequency stability by repeated trials.

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* See Appendix Note # 18

Glossary of Symbols

$B_1(N, r, \mu)$, $B_2(r, \mu)$	Bias function for variances based on finite samples of a process with a power-law spectral density. (See [13].)
C_α	A real constant defined by (70).
c_0, c_1	Real constants.
$c(t)$	A real, deterministic function of time.
$D_x^2(\tau)$	Expected value of the squared second difference of $x(t)$ with lag time τ. See (80).
$f \equiv \omega/2\pi$	Fourier frequency variable.
f_h	High-frequency cutoff of an idealized infinitely sharp cutoff low-pass filter.
f_l	Low-frequency cutoff of an idealized infinitely sharp cutoff, high-pass filter.
$g(t)$	A real function of time.
h_α	Positive real coefficient of f^α in a power series expansion of the spectral density of the function $y(t)$.
i, j, k, m, n	Integers, often a dummy index of summation.
M	Positive integer giving the number of cycles averaged.
N	Positive integer giving the number of data points used in obtaining a sample variance.
$n(t)$	A nondeterministic function of time.
$R_y(\tau)$	Autocovariance function of $y(t)$. See (58).
r	Positive real number defined by $r \equiv T/\tau$.

S	An intermediate term used in deriving (23). The definition of S is given by (64).
$S_g(f)$	One-sided (power) spectral density on a per hertz basis of the pure real function $g(t)$. The dimensions of $S_g(f)$ are the dimensions of $g^2(t)/f$.
$S_y(f)$	A definition for the measure of frequency stability. One-sided (power) spectral density of $y(t)$ on a per hertz basis. The dimensions of $S_y(f)$ are Hz^{-1}.
T	Time interval between the beginnings of two successive measurements of average frequency.
t	Time variable.
t_0	An arbitrary fixed instant of time.
t_k	The time coordinate of the beginning of the kth measurement of average frequency. By definition, $t_{k+1} = t_k + T$, $k = 0, 1, 2 \cdots$.
u	Dummy variable of integration; $u \equiv \pi f \tau$.
$V(t)$	Instantaneous output voltage of signal generator. See (2).
V_0	Nominal peak amplitude of signal generator output. See (2).
$V_r(t)$	Instantaneous voltage of reference signal. See (40).
V_{0r}	Peak amplitude of reference signal. See (40).
$v(t)$	Voltage output of ideal product detector.
$v'(t)$	Low-pass filtered output of product detector.
$x(t)$	Real function of time related to the phase of the signal $V(t)$ by $x(t) \equiv [\varphi(t)]/(2\pi\nu_0)$.
$\hat{x}(t)$	A predicted value for $x(t)$.
$y(t)$	Fractional frequency offset of $V(t)$ from the nominal frequency. See (7).
$\bar{y}_k$	Average fractional frequency offset during the kth measurement interval. See (9).
$\langle \bar{y} \rangle_N$	The sample average of N successive values of $\bar{y}_k$. See (76).
$z_n(t)$	Nondeterministic (noise) function with (power) spectral density given by (25).
α	Exponent of f for a power-law spectral density.
γ	Positive real constant.
$\delta_k(r-1)$	The Kronecker δ function defined by $\delta_k(r-1) \equiv \begin{cases} 1, & \text{if } r = 1 \\ 0, & \text{otherwise.} \end{cases}$
$\epsilon(t)$	Amplitude fluctuations of signal. See (2).
μ	Exponent of τ. See (29).
$\nu(t)$	Instantaneous frequency of $V(t)$. Defined by $$\nu(t) \equiv \frac{1}{2\pi} \frac{d}{dt} \Phi(t).$$
ν_0	Nominal (constant) frequency of $V(t)$.
$\kappa(t)$	The Fourier transform of $n(t)$.
$\sigma_y^2(N, T, \tau)$	Sample variance of N averages of $y(t)$, each of duration τ, and spaced every T units of time. See (10).
$\langle \sigma_y^2(N, T, \tau) \rangle$	Average value of the sample variance $\sigma_y^2(N, T, \tau)$.
$\sigma_y^2(\tau)$	A second choice of the definition for the measure of frequency stability. Defined by $\sigma_y^2(\tau) \equiv \langle \sigma_y^2(N = 2, T = \tau, \tau) \rangle$.
$\sigma_x^2(\tau)$	Time stability measure defined by $\sigma_x^2(\tau) \equiv \tau^2 \sigma_y^2(\tau)$.
τ	Duration of averaging period of $y(t)$ to obtain $\bar{y}_k$. See (9).
$\Phi(t)$	Instantaneous phase of $V(t)$. Defined by $\Phi(t) \equiv 2\pi\nu_0 t + \varphi(t)$.
$\varphi(t)$	Instantaneous phase fluctuations about the ideal phase $2\pi\nu_0 t$. See (2).
$\psi_x^2(T, \tau)$	Mean-square time error for Doppler radar. See (82).
$\omega \equiv 2\pi f$	Angular Fourier frequency variable.

I. Introduction

THE measurement of frequency and fluctuations in frequency has received such great attention for so many years that it is surprising that the concept of frequency stability does not have a universally accepted definition. At least part of the reason has been that some uses are most readily described in the frequency domain and other uses in the time domain, as well as in combinations of the two. This situation is further complicated by the fact that only recently have noise models been presented that both adequately describe performance and allow a translation between the time and frequency domains. Indeed, only recently has it been recognized that there can be a wide discrepancy between commonly used time domain measures themselves. Following the NASA–IEEE Symposium on Short-Term Stability in 1964 and the Special Issue on Frequency Stability in the PROCEEDINGS OF THE IEEE, February 1966, it now seems reasonable to propose a definition of frequency stability. The present paper is presented as technical background for an eventual IEEE standard definition.

This paper attempts to present (as concisely as practical) adequate, self-consistent definitions of frequency stability. Since more than one definition of frequency stability is presented, an important part of this paper (perhaps the most important part) deals with translations among the suggested definitions of frequency stability. The applicability of these definitions to the more common noise models is demonstrated.

Consistent with an attempt to be concise, the references cited have been selected on the basis of being of most value to the reader rather than on the basis of being exhaustive. An exhaustive reference list covering the subject of frequency stability would itself be a voluminous publication.

Almost any signal generator is influenced to some extent by its environment. Thus observed frequency instabilities may be traced, for example, to changes in ambient temperature, supply voltages, magnetic field, barometric pressure, humidity, physical vibration, or even output loading, to mention the more obvious. While these environmental influences may be extremely important for many applications, the definition of frequency stability presented here is independent of these causal factors. In effect, we cannot hope to present an exhaustive list of environmental factors and a prescription for handling each even though, in some cases, these environmental factors may be by far the most important. Given a particular signal generator in a particular environment, one can obtain its frequency stability with the measures presented below, but one should not then expect an accurate prediction of frequency stability in a new environment.

It is natural to expect any definition of stability to involve various statistical considerations such as stationarity, ergodicity, average, variance, spectral density, etc. There often exist fundamental difficulties in rigorous attempts to bring these concepts into the laboratory. It is worth considering, specifically, the concept of stationarity since it is a concept at the root of many statistical discussions.

A random process is mathematically defined as stationary if every translation of the time coordinate maps the ensemble onto itself. As a necessary condition, if one looks at the ensemble at one instant of time t, the distribution in values within the ensemble is exactly the same as at any other instant of time t'. This is not to imply that the elements of the ensemble are constant in time, but, as one element changes value with time, other elements of the ensemble assume the previous values. Looking at it in another way, by observing the ensemble at some instant of time, one can deduce no information as to when the particular instant was chosen. This same sort of invariance of the *joint* distribution holds for any set of times $t_1, t_2, \cdots, t_n$ and its translation $t_1 + \tau, t_2 + \tau, \cdots, t_n + \tau$.

It is apparent that any ensemble that has a finite past as well as a finite future cannot be stationary, and this neatly excludes the real world and anything of practical interest. The concept of stationarity does violence to concepts of causality since we implicitly feel that current performance (i.e., the applicability of stationary statistics) cannot be logically dependent upon future events (i.e., if the process is terminated some time in the distant future). Also, the verification of stationarity would involve hypothetical measurements that are *not* experimentally feasible, and therefore the concept of stationarity is not directly relevant to experimentation.

Actually the utility of statistics is in the formation of idealized models that *reasonably* describe significant observables of real systems. One may, for example, consider a hypothetical ensemble of noises with certain properties (such as stationarity) as a model for a particular real device. If a model is to be acceptable, it should have at least two properties: first, the model should be tractable; that is, one should be able to easily arrive at estimates for the elements of the models; and second, the model should be consistent with *observables* derived from the real device that it is simulating.

Notice that one does not need to know that the device was selected from a stationary ensemble, but only that the observables derived from the device are *consistent* with, say, elements of a hypothetically stationary ensemble. Notice also that the actual model used may depend upon how clever the experimenter–theorist is in generating models.

It is worth noting, however, that while some texts on statistics give "tests for stationarity," these tests are almost always inadequate. Typically, these tests determine only if there is a substantial fraction of the noise power in Fourier frequencies whose periods are of the same order as the data length or longer. While this may be very important, it is *not* logically essential to the concept of stationarity. If a nonstationary model actually becomes common, it will almost surely be because it is useful or convenient and not because the process is "actually nonstationary." Indeed, the phrase "actually nonstationary" appears to have no meaning in an operational sense. In short, stationarity (or nonstationarity) is a property of models, *not* a property of data [1].

Fortunately, many statistical models exist that adequately describe most present-day signal generators; many of these models are considered below. It is obvious that one cannot guarantee that all signal generators are adequately described by these models, but the authors do feel they are adequate for the description of most signal generators presently encountered.

II. Statement of the Problem

To be useful, a measure of frequency stability must allow one to predict performance of signal generators used in a wide variety of situations as well as allow one to make meaningful relative comparisons among signal generators. One must be able to predict performance in devices that may most easily be described either in the time domain, or in the frequency domain, or in a combination of the two. This prediction of performance may involve actual distribution functions, and thus second moment measures (such as power spectra and variances) are not totally adequate.

Two common types of equipment used to evaluate the performance of a frequency source are (analog) spectrum analyzers (frequency domain) and digital electronic counters (time domain). On occasion the digital counter data are converted to power spectra by computers. One must realize that any piece of equipment simultaneously has certain aspects most easily described in the time domain and other aspects most easily described in the frequency domain. For example, an electronic counter has a high-frequency limitation, an experimental spectra are determined with finite time averages.

Research has established that ordinary oscillators demonstrate noise, which appears to be a superposition of causally generated signals and random nondeterministic noises. The random noises include thermal noise, shot noise, noises of undetermined origin (such as flicker noise), and integrals of these noises.

One might well expect that for the more general cases one would need to use a nonstationary model (not stationary even in the wide sense, i.e., the covariance sense). Nonstationarity would, however, introduce significant difficulties in the passage between the frequency and time domains. It is interesting to note that, so far, experimenters have seldom found a nonstationary (covariance) model useful in describing actual oscillators.

In what follows, an attempt has been made to separate general statements that hold for any noise or perturbation from the statements that apply only to specific models. It is important that these distinctions be kept in mind.

III. Background and Definitions

To discuss the concept of frequency stability immediately implies that frequency can change with time and thus one is not considering Fourier frequencies (at least at this point). The conventional definition of instantaneous (angular) frequency is the time rate of change of phase; that is

$$2\pi\nu(t) \equiv \frac{d\Phi(t)}{dt} \equiv \dot{\Phi}(t) \qquad (1)$$

where $\Phi(t)$ is the instantaneous phase of the oscillator. This paper uses the convention that time-dependent frequencies of oscillators are denoted by $\nu(t)$ (cycle frequency, hertz), and Fourier frequencies are denoted by ω (angular frequency) or f (cycle frequency, hertz) where $\omega \equiv 2\pi f$. In order for (1) to have meaning, the phase $\Phi(t)$ must be a well-defined function. This restriction immediately eliminates some "nonsinusoidal" signals such as a pure random uncorrelated ("white") noise. For most real signal generators, the concept of phase is reasonably amenable to an operational definition and this restriction is not serious.

Of great importance to this paper is the concept of spectral density, $S_g(f)$. The notation $S_g(f)$ is to represent the one-sided spectral density of the (pure real) function $g(t)$ on a per hertz basis; that is, the total "power" or mean-square value of $g(t)$ is given by

$$\int_0^\infty S_g(f)\, df.$$

Since the spectral density is such an important concept to what follows, it is worthwhile to present some important references on spectrum estimation. There are many references on the estimation of spectra from data records, but worthy of special note are [2]–[5].

IV. Definition of Measures of Frequency Stability (Second-Moment Type)

A. General

Consider a signal generator whose instantaneous output voltage $V(t)$ may be written as

$$V(t) = [V_0 + \epsilon(t)] \sin [2\pi\nu_0 t + \varphi(t)] \qquad (2)$$

where V_0 and ν_0 are the nominal amplitude and frequency, respectively, of the output and it is assumed that

$$\left|\frac{\epsilon(t)}{V_0}\right| \ll 1 \qquad (3)$$

and

$$\left|\frac{\dot{\varphi}(t)}{2\pi\nu_0}\right| \ll 1 \qquad (4)$$

for substantially all time t. Making use of (1) and (2) one sees that

$$\Phi(t) = 2\pi\nu_0 t + \varphi(t) \qquad (5)$$

and

$$\nu(t) = \nu_0 + \frac{1}{2\pi}\dot{\varphi}(t). \qquad (6)$$

Equations (3) and (4) are essential in order that $\varphi(t)$ may be defined conveniently and unambiguously (see measurement section).

Since (4) must be valid even to speak of an instantaneous frequency, there is no real need to distinguish stability measures from instability measures. That is, any fractional frequency stability measure will be far from unity, and the chance of confusion is slight. It is true that in a very strict sense people usually measure instability and speak of stability. Because the chances of confusion are so slight, the authors have chosen to continue in the custom of measuring "instability" and speaking of stability (a number always much less than unity).

Of significant interest to many people is the radio frequency (RF) spectral density $S_V(f)$. This is of direct concern in spectroscopy and radar. However, this is *not* a good primary measure of frequency stability for two reasons. First, fluctuations in the amplitude $\epsilon(t)$ contribute directly to $S_V(f)$; and second, for many cases when

$\epsilon(t)$ is insignificant, the RF spectrum $S_V(f)$ is not uniquely related to the frequency fluctuations [6].

B. General: First Definition of the Measure of Frequency Stability—Frequency Domain

By definition, let

$$y(t) \equiv \frac{\dot{\varphi}(t)}{2\pi\nu_0}, \qquad (7)$$

where $\varphi(t)$ and ν_0 are as in (2). Thus $y(t)$ is the instantaneous fractional frequency deviation from the nominal frequency ν_0. A proposed definition of frequency stability is the spectral density $S_y(f)$ of the instantaneous fractional frequency fluctuations $y(t)$. The function $S_y(f)$ has the dimensions of Hz^{-1}.

One can show [7] that if $S_\varphi(f)$ is the spectral density of the phase fluctuations, then

$$S_y(f) = \left(\frac{1}{2\pi\nu_0}\right)^2 S_{\dot{\varphi}}(f) = \left(\frac{1}{\nu_0}\right)^2 f^2 S_\varphi(f). \qquad (8)$$

Thus a knowledge of the spectral density of the phase fluctuations $S_\varphi(f)$ allows a knowledge of the spectral density of the frequency fluctuations $S_y(f)$, the first definition of frequency stability. Of course, $S_y(f)$ cannot be *perfectly* measured—this is the case for any physical quantity; useful estimates of $S_y(f)$ are, however, easily obtainable.

C. General: Second Definition of the Measure of Frequency Stability—Time Domain

The second definition is based on the sample variance of the fractional frequency fluctuations. In order to present this measure of frequency stability, define $\bar{y}_k$ by the relation

$$\bar{y}_k \equiv \frac{1}{\tau}\int_{t_k}^{t_k+\tau} y(t)\,dt = \frac{\varphi(t_k+\tau) - \varphi(t_k)}{2\pi\nu_0\tau}, \qquad (9)$$

where $t_{k+1} = t_k + T$, $k = 0, 1, 2, \cdots$, T is the repetition interval for measurements of duration τ, and t_0 is arbitrary. Conventional frequency counters measure the number of cycles in a period τ; that is, they measure $\nu_0\tau(1 + \bar{y}_k)$. When τ is 1 s they count the number of $\nu_0(1 + \bar{y}_k)$. The second measure of frequency stability, then, is defined in analogy to the sample variance by the relation

$$\langle \sigma_y^2(N, T, \tau)\rangle \equiv \left\langle \frac{1}{N-1}\sum_{n=1}^{N}\left(\bar{y}_n - \frac{1}{N}\sum_{k=1}^{N}\bar{y}_k\right)^2\right\rangle, \qquad (10)$$

where $\langle g\rangle$ denotes the infinite time average of g. This measure of frequency stability is dimensionless.

In many situations it would be wrong to assume that (10) converges to a meaningful limit as $N \to \infty$. First, of course, one cannot practically let N approach infinity and, second, it is known that some actual noise processes contain substantial fractions of the total noise power in the Fourier frequency range below one cycle per year. In order to improve comparability of data, it is important to specify particular N and T. For the preferred definition we recommend choosing $N = 2$ and $T = \tau$ (i.e., no dead time between measurements). Writing $\langle\sigma_y^2(N=2, T=\tau, \tau)\rangle$ as $\sigma_y^2(\tau)$, the Allan variance [8], the proposed measure of frequency stability in the time domain may be written as

$$\sigma_y^2(\tau) = \left\langle \frac{(\bar{y}_{k+1} - \bar{y}_k)^2}{2}\right\rangle \qquad (11)$$

for $T = \tau$.

Of course, the experimental estimate of $\sigma_y^2(\tau)$ must be obtained from finite samples of data, and one can never obtain perfect confidence in the estimate; the true time average is not realizable in a real situation. One estimates $\sigma_y^2(\tau)$ from a finite number (say, m) of values of $\sigma_y^2(2, \tau, \tau)$ and averages to obtain an estimate of $\sigma_y^2(\tau)$. Appendix I shows that the ensemble average of $\sigma_y^2(2, \tau, \tau)$ is convergent (i.e., as $m \to \infty$) even for noise processes that do not have convergent $\langle\sigma_y^2(N, \tau, \tau)\rangle$ as $N \to \infty$. Therefore, $\sigma_y^2(\tau)$ has greater utility as an idealization than does $\langle\sigma_y^2(\infty, \tau, \tau)\rangle$ even though both involve assumptions of infinite averages. In effect, increasing N causes $\sigma_y^2(N, T, \tau)$ to become more sensitive to the low-frequency components of $S_y(f)$. In practice, one must distinguish between an experimental estimate of a quantity (say, of $\sigma_y^2(\tau)$) and its idealized value. It is reasonable to believe that extensions to the concept of statistical ("quality") control [9] may prove useful here. One should, of course, specify the actual number m of independent samples used for an estimate of $\sigma_y^2(\tau)$.

In summary, therefore, $S_y(f)$ is the proposed measure of (instantaneous) frequency stability in the (Fourier) frequency domain and $\sigma_y^2(\tau)$ is the proposed measure of frequency stability in the time domain.

D. Distributions

It is natural that people first become involved with second moment measures of statistical quantities and only later with actual distributions. This is certainly true with frequency stability. While one can specify the argument of a distribution function to be, say $(\bar{y}_{k+1} - \bar{y}_k)$, it makes sense to postpone such a specfication until a real use has materialized for a particular distribution function. This paper does not attempt to specify a preferred distribution function for frequency fluctuations.

E. Treatment of Systematic Variations

1) General: The definition of frequency stability $\sigma_y^2(\tau)$ in the time domain is useful for many situations. However, some oscillators, for example, exhibit an aging or almost linear drift of frequency with time. For some applications, this trend may be calculated and should be removed [8] before estimating $\sigma_y^2(\tau)$.

In general, a systematic trend is perfectly deterministic (i.e., predictable) while the noise is nondeterministic. Consider a function $g(t)$, which may be written in the form

$$g(t) = c(t) + n(t) \tag{12}$$

where $c(t)$ is some deterministic function of time and $n(t)$, the noise, is a nondeterministic function of time. We will define $c(t)$ to be the *systematic trend* to the function $g(t)$. A problem of significance here is to determine when and in what sense $c(t)$ is measurable.

2) Specific Case—Linear Drift: As an example, if we consider a typical quartz crystal oscillator whose fractional frequency deviation is $y(t)$, we may let

$$g(t) = \frac{d}{dt} y(t). \tag{13}$$

With these conditions, $c(t)$ is the drift rate of the oscillator (e.g., 10^{-10}/day) and $n(t)$ is related to the frequency "noise" of the oscillator by a time derivative. One sees that the time average of $g(t)$ becomes

$$\frac{1}{T}\int_{t_0}^{t_0+T} g(t)\,dt = c_1 + \frac{1}{T}\int_{t_0}^{t_0+T} n(t)\,dt \tag{14}$$

where $c(t) = c_1$ is assumed to be the constant drift rate of the oscillator. In order for c_1 to be an observable, it is natural to expect the average of the noise term to vanish, that is, converge to zero.

It is instructive to assume [8], [10] that in addition to a linear drift, the oscillator is perturbed by a flicker noise, i.e.,

$$S_y(f) = \begin{cases} h_{-1} f^{-1}, & 0 < f \le f_h \\ 0, & f > f_h \end{cases} \tag{15}$$

where h_{-1} is a constant (see Section V-A-2) and thus,

$$S_n(f) = \begin{cases} (2\pi)^2 h_{-1} f, & 0 \le f \le f_h \\ 0, & f > f_h \end{cases} \tag{16}$$

for the oscillator we are considering. With these assumptions, it is seen that

$$\lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{t_0+T} n(t)\,dt = \kappa(0) = 0 \tag{17}$$

and that

$$\lim_{T\to\infty} \left\{ \text{variance} \left[\frac{1}{T}\int_{t_0}^{t_0+T} n(t)\,dt \right] \right\} = 0 \tag{18}$$

where $\kappa(f)$ is the fourier transform of $n(t)$. Since $S_n(0) = 0$, $\kappa(0)$ must also vanish both in probability and in mean square. Thus, not only does $n(t)$ average to zero, but one may obtain arbitrarily good confidence on the result by longer averages.

Having shown that one can reliably estimate the drift rate c_1 of this (common) oscillator, it is instructive to attempt to fit a straight line to the frequency aging. That is, let

$$g(t) = y(t) \tag{19}$$

and thus

$$g(t) = c_0 + c_1(t - t_0) + n'(t) \tag{20}$$

where c_0 is the frequency intercept at $t = t_0$ and c_1 is the drift rate previously determined. A problem arises here because

$$S_{n'}(f) = S_y(f) \tag{21}$$

and

$$\lim_{T\to\infty} \left\{ \text{variance} \left[\frac{1}{T}\int_{t_0}^{t_0+T} n'(t)\,dt \right] \right\} = \infty \tag{22}$$

for the noise model we have assumed. This follows from the fact that the (infinite N) variance of a flicker noise process is infinite [7], [8], [10]. Thus, c_0 cannot be measured with any realistic precision, at least, in an absolute sense.

We may interpret these results as follows. After experimenting with the oscillator for a period of time one can fit an empirical equation to $y(t)$ of the form

$$y(t) = c_0 + tc_1 + n'(t),$$

where $n'(t)$ is nondeterministic. At some later time it is possible to reevaluate the coefficients c_0 and c_1. According to what has been said, the drift rate c_1 should be reproducible to within the confidence estimates of the experiment regardless of when it is reevaluated. For c_0, however, this is not true. In fact, the more one attempts to evaluate c_0, the larger the fluctuations are in the result.

Depending on the spectral density of the noise term, it may be possible to predict future measurements of c_0 and to place realistic confidence limits on the prediction [11]. For the case considered here, however, these confidence limits tend to infinity when the prediction interval is increased. Thus, in a certain sense, c_0 is "measurable" but it is not in statistical control (to use the language of the quality control engineer [9]).

V. Translations Among Frequency Stability Measures

A. Frequency Domain to Time Domain

1) General: It is of value to define $r = T/\tau$; that is, r is the ratio of the time interval between successive measurements to the duration of the averaging period. Cutler has shown (see Appendix I) that

$$\langle \sigma_y^2(N, T, \tau) \rangle$$ *

$$= \frac{N}{(N-1)} \int_0^\infty df\, S_y(f) \frac{[\sin^2(\pi f \tau)]}{(\pi f \tau)^2} \left\{ 1 - \frac{\sin^2(\pi r f N \tau)}{N^2 \sin^2(\pi r f \tau)} \right\}. \tag{23}$$

Equation (23) in principle allows one to calculate the time-domain stability $\langle \sigma_y^2(N, T, \tau) \rangle$ from the frequency-domain stability $S_y(f)$.

2) Specific Model: A model that has been found useful [8], [10]–[13] consists of a set of five independent noise processes $z_n(t)$, $n = -2, -1, 0, 1, 2$, such that

* See Appendix Note # 19

$$y(t) = z_{-2}(t) + z_{-1}(t) + z_0(t) + z_1(t) + z_2(t) \quad (24)$$

and the spectral density of z_n is given by

$$S_{z_n}(f) = \begin{cases} h_n f^n, & 0 \le f \le f_h \\ 0, & f > f_h, n = -2, -1, 0, 1, 2, \end{cases} \quad (25)$$

where the h_n are constants. Thus, $S_y(f)$ becomes

$$S_y(f) = h_{-2}f^{-2} + h_{-1}f^{-1} + h_0 + h_1 f + h_2 f^2, \quad (26)$$

for $0 \le f \le f_h$ and $S_y(f)$ is assumed to be negligible beyond this range. In effect, each z_n contributes to both $S_y(f)$ and $\langle \sigma_y^2(N, T, \tau)\rangle$ independently of the other z_n. The contributions of the z_n to $\langle \sigma_y^2(N, T, \tau)\rangle$ are tabulated in Appendix II.

Any electronic device has a finite bandwidth and this certainly applies to frequency-measuring equipment also. For fractional frequency fluctuations $y(t)$ whose spectral density varies as

$$S_y(f) \sim f^\alpha, \quad \alpha \ge -1 \quad (27)$$

for the higher Fourier components, one sees (from Appendix I) that $\langle \sigma_y^2(N, T, \tau)\rangle$ may depend on the exact shape of the frequency cutoff. This is true because a substantial fraction of the noise "power" may be in these higher Fourier components. As a simplifying assumption, this paper assumes a sharp cutoff in noise "power" at the frequency f_h for the noise models. It is apparent from the tables of Appendix II that the time domain measure of frequency stability may depend on f_h in a very important way, and, in some practical cases, the actual shape of the frequency cutoff may be very important [7]. On the other hand, there are many practical measurements where the value of f_h has little or no effect. Good practice, however, dictates that the system noise bandwidth f_h should be specified with any results.

In actual practice, the model of (24)–(26) seems to fit almost all real frequency sources. Typically, only two or three of the h-coefficients are actually significant for a real device and the others can be neglected. Because of its applicability, this model is used in much of what follows. Since the z_n are assumed to be independent noises, it is normally sufficient to compute the effects for a general z_n and recognize that the superposition can be accomplished by simple additions for their contributions to $S_y(f)$ or $\langle \sigma_y^2(N, T, \tau)\rangle$.

B. Time Domain to Frequency Domain

1) General: For general $\langle \sigma_y^2(N, T, \tau)\rangle$ no simple prescription is available for translation into the frequency domain. For this reason, one might prefer $S_y(f)$ as a general measure of frequency stability. This is especially true for theoretical work.

2) Specific Model: Equations (24)–(26) form a realistic model that fits the random nondeterministic noises found on most signal generators. Obviously, if this is a good model, then the tables in Appendix II may be used (in reverse) to translate into the frequency domain.

Allan [8] and Vessot [12] showed that if

$$S_y(f) = \begin{cases} h_\alpha f^\alpha, & 0 \le f \le f_h \\ 0, & f > f_h \end{cases} \quad (28)$$

where α is a constant, then

$$\langle \sigma_y^2(N, T, \tau)\rangle \sim |\tau|^\mu, \quad 2\pi\tau f_h \gg 1 \quad (29)$$

for N and $r = T/\tau$ held constant. The constant μ is related to α by the mapping shown[1] in Fig. 1. If (28) and (29) hold over a reasonable range for a signal generator, then (28) can be substituted into (23) and evaluated to determine the constant h_α from measurements of $\langle \sigma_y^2(N, T, \tau)\rangle$. It should be noted that the model of (28) and (29) may be easily extended to a superposition of similar noises as in (26).

C. Translations Among the Time-Domain Measures

1) General: Since $\langle \sigma_y^2(N, T, \tau)\rangle$ is a function of N, T, and τ (for some types of noise f_h is also important), it is very desirable to be able to translate among different sets of N, T, and τ (f_h held constant). This is, however, not possible in general.

2) Specific Model: It is useful to restrict consideration to a case described by (28) and (29). Superpositions of independent noises with different power-law types of spectral densities (i.e., different α) can also be treated by this technique, e.g., (26). One may define two "bias functions," B_1 and B_2 by the relations [13]

$$B_1(N, r, \mu) \equiv \frac{\langle \sigma_y^2(N, T, \tau)\rangle}{\langle \sigma_y^2(2, T, \tau)\rangle} \quad (30)$$

and

$$B_2(r, \mu) \equiv \frac{\langle \sigma_y^2(2, T, \tau)\rangle}{\langle \sigma_y^2(2, \tau, \tau)\rangle} \quad (31)$$

where $r \equiv T/\tau$ and μ is related to α by the mapping of Fig. 1. In words, B_1 is the ratio of the average variance for N samples to the average variance for two samples (everything else held constant), while B_2 is the ratio of the average variance with dead time between measurements ($r \ne 1$) to that of no dead time ($r = 1$ and with $N = 2$ and τ held constant). These functions are tabulated in [13]. Figs. 2 and 3 show a computer plot of $B_1(N, r = 1, \mu)$ and $B_2(r, \mu)$.

Suppose one has an experimental estimate of $\langle \sigma_y^2(N_1, T_1, \tau_1)\rangle$ and its spectral type is known, i.e., (28) and (29) form a good model and μ is known. Suppose also that one wishes to know the variance at some other set of measurement parameters N_2, T_2, τ_2. An unbiased estimate of $\langle \sigma_y^2(N_2, T_2, \tau_2)\rangle$ may be calculated by

[1] It should be noted that in Allan [8], the exponent α corresponds to the spectrum of *phase* fluctuations while variances are taken over average *frequency* fluctuations. In the present paper, α is identical to the exponent $\alpha + 2$ in [8].

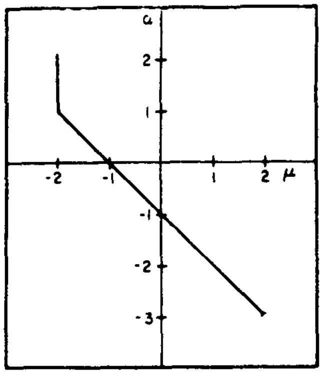

Fig. 1. $\mu - \alpha$ mapping.

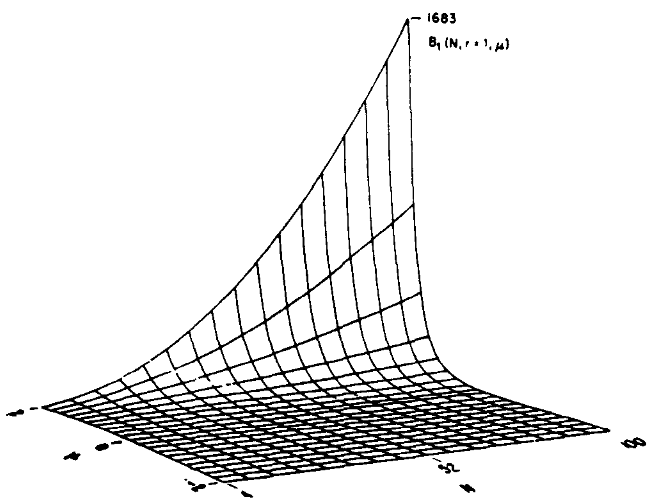

Fig. 2. Function $B_1(N, r = 1, \mu)$.

$$\langle\sigma_y^2(N_2, T_2, \tau_2)\rangle = \left(\frac{\tau_2}{\tau_1}\right)^\mu \cdot \left[\frac{B_1(N_2, r_2, \mu)B_2(r_2, \mu)}{B_1(N_1, r_1, \mu)B_2(r_1, \mu)}\right]\langle\sigma_y^2(N_1, T_1, \tau_1)\rangle, \qquad (32)$$

where $r_1 = T_1/\tau_1$ and $r_2 = T_2/\tau_2$.

3) General: While it is true that the concept of the bias functions B_1 and B_2 could be extended to other processes besides those with the power-law types of spectral densities, this generalization has not been done. Indeed, spectra of the form given in (28) [or superpositions of such spectra as in (26)] seem to be the most common types of nondeterministic noises encountered in signal generators and associated equipment. For other types of fluctuations (such as causally generated perturbations), translations must be handled on an individual basis.

VI. Applications of Stability Measures

Obviously, if one of the stability measures is exactly the important parameter in the use of a signal generator, the stability measure's application is trivial. Some nontrivial applications arise when one is interested in a different parameter, such as in the use of an oscillator in Doppler radar measurements or in clocks.

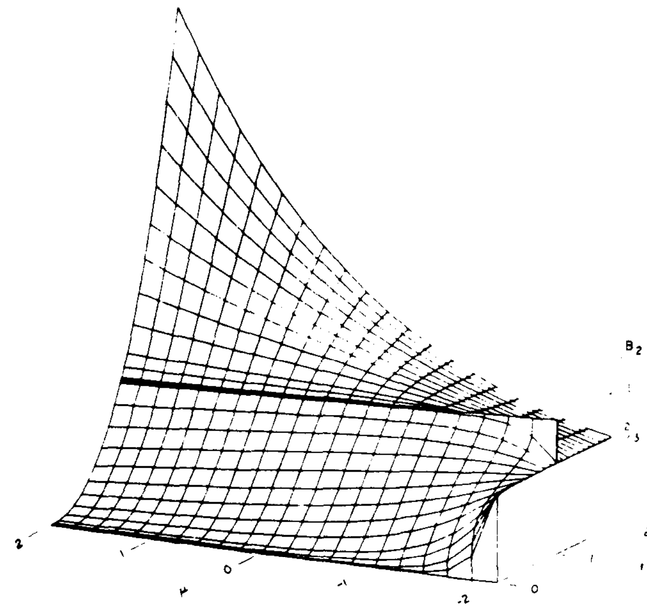

Fig. 3. Bias function $B_2(r, \mu)$.

A. Doppler Radar

1) General: From its transmitted signal, a Doppler radar receives from a moving target a frequency-shifted return signal in the presence of other large signals. These large signals can include clutter (ground return) and transmitter leakage into the receiver (spillover). Instabilities of radar signals result in noise energy on the clutter return, on spillover, and on local oscillators in the equipment.

The limitations of subclutter visibility (SCV) rejections due to the radar signals themselves are related to the RF power spectral density $S_V(f)$. The quantity typically referred to is the carrier-to-noise ratio and can be mathematically approximated by the quantity

$$\frac{S_V(f)}{\int_0^\infty S_V(f')df'}.$$

The effects of coherence of target return and other radar parameters are amply considered in the literature [14]–[17].

2) Special Case: Because FM effects generally predominate over AM effects, this carrier-to-noise ratio is approximately given by [6]

$$\frac{S_V(f)}{\int_0^\infty S_V(f')\,df'} \approx \tfrac{1}{2}S_\varphi(|f - \nu_0|), \qquad (33)$$

for many signal sources provided $|f - \nu_0|$ is sufficiently greater than zero. (The factor of $\frac{1}{2}$ arises from the fact that $S_\varphi(f)$ is a one-sided spectrum.) Thus, if $f - \nu_0$ is a frequency separation from the carrier, the carrier-to-noise ratio at that point is approximately

$$\tfrac{1}{2}S_\varphi(|f-\nu_0|) = \frac{1}{2}\left(\frac{\nu_0}{f-\nu_0}\right)^2 S_\nu(|f-\nu_0|). \quad (34)$$

B. Clock Errors

1) General: A clock is a device that counts the cycles of a periodic phenomenon. Thus, the reading error $x(t)$ of a clock run from the signal given by (2) is

$$x(t) = \frac{\varphi(t)}{2\pi\nu_0} \quad (35)$$

and the dimensions of $x(t)$ are seconds.

If this clock is a secondary standard, then one could have available some past history of $x(t)$, the time error relative to the standard clock. It often occurs that one is interested in predicting the clock error $x(t)$ for some future date, say $t_0+\tau$, where t_0 is the present date. Obviously, this is a problem in pure prediction and can be handled by conventional methods [3].

2) Special Case: Although one could handle the prediction of clock errors by the rigorous methods of prediction theory, it is more common to use simpler prediction methods [10], [11]. In particular, one often predicts a clock error for the future by adding to the present error a correction that is derived from the current rate of gain (or loss) of time. That is, the predicted error $\hat{x}(t_0+\tau)$ is related to the past history of $x(t)$ by

$$\hat{x}(t_0+\tau) = x(t_0) + T\left[\frac{x(t_0)-x(t_0-T)}{T}\right]. \quad (36)*$$

It is typical to let $T=\tau$.

Thus, the mean-square error of prediction for $T=\tau$ becomes

$$\langle[x(t_0+\tau)-\hat{x}(t_0+\tau)]^2\rangle = \langle[x(t_0+\tau)-2x(t_0)+x(t_0-\tau)]^2\rangle, \quad (37)$$

which, with the aid of (11), can be written in the form

$$\langle[x(t_0+\tau)-\hat{x}(t_0+\tau)]^2\rangle = 2\tau^2\sigma_y^2(\tau). \quad (38)$$

One can define a *time* stability measure $\sigma_x^2(\tau)$ by

$$\sigma_x^2(\tau) \equiv \tau^2\sigma_y^2(\tau). \quad (39)$$

Clearly, however, the actual errors of prediction of clock readings are dependent on the prediction algorithm used and the utility of such a definition as $\sigma_x^2(\tau)$ is not great. Caution should be used in employing this definition.

VII. Measurement Techniques for Frequency Stability

A. Heterodyne Techniques (General)

It is possible for oscillators to be very stable and values of $\sigma_y(\tau)$ can be as small as 10^{-14} in some state-of-the-art equipment. Thus, one often needs measuring techniques capable of resolving very small fluctuations in $y(t)$. One of the most common techniques is a heterodyne or beat frequency technique. In this method, the signal from the oscillator to be tested is mixed with a reference signal of almost the same frequency as the test oscillator in order that one is left with a lower average frequency for analysis without reducing the frequency (or phase) fluctuations themselves. Following Vessot *et al.* [18], consider an ideal reference oscillator whose output signal is

$$V_r(t) = V_{0r}\sin 2\pi\nu_0 t \quad (40)$$

and a second oscillator whose output voltage $V(t)$ is given by (2): $V(t) = [V_0+\epsilon(t)]\sin[2\pi\nu_0 t+\varphi(t)]$. Let these two signals be mixed in a product detector; that is, the output of the product detector $v(t)$ is equal to the product $\gamma V(t)\times V_r(t)$, where γ is a constant (see Fig. 4).

Let $v(t)$, in turn, be processed by a sharp low-pass filter with cutoff frequency f_h' such that

$$0 < f_h < f_h' < \nu_0. \quad (41)$$

One may write

$$\begin{aligned}\gamma V(t)\cdot V_r(t) &= \gamma V_{0r}(V_0+\epsilon)[\sin 2\pi\nu_0 t][\sin(2\pi\nu_0 t+\varphi)]\\ &= v(t) = \gamma\frac{(V_{0r}V_0)}{2}\left(1+\frac{\epsilon}{V_0}\right)[\cos\varphi - \cos(4\pi\nu_0 t+\varphi)]. \quad (42)\end{aligned}$$

Assume that $\cos[\varphi(t)]$ has essentially no power in Fourier frequencies f in the region $f \ge f_h'$. The effect of the low-pass filter then is to remove the second term on the extreme right of (42); that is

$$v'(t) = \gamma\frac{V_{0r}V_0}{2}\left(1+\frac{\epsilon}{V_0}\right)\cos\varphi(t). \quad (43)$$

This separation of terms by the filter is correct only if $|[\dot{\varphi}(t)/2\pi\nu_0]| \ll 1$ for all t (4).

The following two cases are of interest.

Case I: The relative phase of the oscillators is adjusted so that $|\varphi(t)| \ll 1$ (in-phase condition) during the period of measurement. Under these conditions

$$v'(t) \approx \frac{\gamma}{2}V_{0r}V_0 + \frac{\gamma}{2}V_{0r}\epsilon(t), \quad (44)$$

since $\cos\varphi(t) \approx 1$. That is to say one detects the amplitude noise $\epsilon(t)$ of the signal.

Case II: The relative phase of the oscillators is adjusted to be in approximate quadrature; that is

$$\varphi'(t) = \varphi(t) + \frac{\pi}{2} \quad (45)$$

where $|\varphi'(t)| \ll 1$. Under these conditions,

$$\cos\varphi(t) = \sin\varphi'(t) \approx \varphi'(t) \quad (46)$$

and

$$v'(t) = \frac{\gamma}{2}V_{0r}V_0\varphi'(t) + \frac{\gamma}{2}V_{0r}\varphi'(t)\epsilon(t). \quad (47)$$

* See Appendix Note # 20

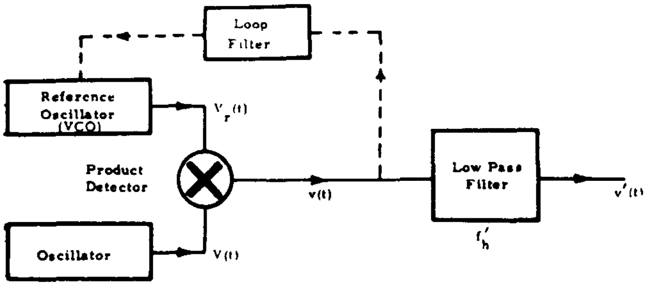

Fig. 4. Heterodyne scheme.

If it is true that $|[\epsilon(t)/V_0]| \ll 1$ for all t (3), then (47) becomes

$$v'(t) \approx \frac{\gamma}{2} V_{0r} V_0 \varphi'(t); \tag{48}$$

that is, $v'(t)$ is proportional to the phase fluctuations. Thus, in order to observe $\varphi'(t)$ by this method, (3) and (4) must be valid. For different average phase values, mixtures of amplitude and phase noise are observed.

In order to maintain the two signals in quadrature for long observational periods, the reference oscillator can be a voltage-controlled oscillator (VCO) and one may feed back the phase error voltage as defined in (48) to control the frequency of the VCO [19]. In this condition of the phase-locked oscillator, the voltage $v'(t)$ is the analog of the *phase* fluctuations for Fourier frequencies above the loop cutoff frequency of the locked loop. For Fourier frequencies below the loop cutoff frequency of the loop, $v'(t)$ is the analog of *frequency* fluctuations. In practice, one should measure the complete servo-loop response.

B. Period Measurement

Assume one has an oscillator whose voltage output may be represented by (2). If $|[\epsilon(t)/V_0]| \ll 1$ for all t and the total phase

$$\Phi(t) = 2\pi\nu_0 t + \varphi(t) \tag{5}$$

is a monotonic function of time (that is, $|[\dot{\varphi}(t)/2\pi\nu_0]| \leq 1$), then the time t between successive positive going zero crossings of $V(t)$ is related to the average frequency during the interval τ. Specifically

$$\frac{1}{\tau} = \nu_0(1 + \bar{y}_n). \tag{49}$$

If one lets τ be the time between a positive going zero crossing of $V(t)$ and the Mth successive positive going zero crossing, then

$$\frac{M}{\tau} = \nu_0(1 + \bar{y}_n). \tag{50}$$

If the variations $\Delta\tau$ of the period are small compared to the average period τ_0, Cutler and Searle [7] have shown that one may make a reasonable approximation to $\langle\sigma_y^2(N, T, \tau_0)\rangle$ using period measurements.

C. Period Measurement With Heterodyning

Suppose that $\varphi(t)$ is a monotonic function of time. The output of the filter of Section VII-A (43) becomes

$$v'(t) \approx \gamma \frac{V_{0r} V_0}{2} \cos \varphi(t) \tag{51}$$

if $|[\epsilon(t)/V_0]| \ll 1$. Then one may measure the period τ of two successive positive zero crossings of $v'(t)$. Thus

$$\frac{1}{\tau} = \nu_0 \, |\bar{y}_n| \tag{52}$$

and for the Mth positive crossover

$$\frac{M}{\tau} = \nu_0 \, |\bar{y}_n|. \tag{53}$$

The magnitude bars appear because $\cos \varphi(t)$ is an even function of $\varphi(t)$. It is impossible to determine by this method alone whether φ is increasing with time or decreasing with time. Since $\bar{y}_n$ may be very small ($\sim 10^{-11}$ or 10^{-12} for very good oscillators), τ may be quite long and thus measurable with a good relative precision.

If the phase $\varphi(t)$ is not monotonic, the true $\bar{y}_n$ may be near zero but one could still have many zeros of $\cos \varphi(t)$ and thus (52) and (53) would not be valid.

D. Frequency Counters

Assume the phase (either Φ or φ) is a montonic function of time. If one counts the number M of positive going zero crossings in a period of time τ, then the average frequency of the signal is M/τ. If we assume that the signal is $V(t)$ as defined in (2), then

$$\frac{M}{\tau} = \nu_0(1 + \bar{y}_n). \tag{54}$$

If we assume that the signal is $v'(t)$ as defined in (48), then

$$\frac{M}{\tau} = \nu_0 \, |\bar{y}_n|. \tag{55}$$

Again, one measures only positive frequencies.

E. Frequency Discriminators

A frequency discriminator is a device that converts frequency fluctuations into an analog voltage by means of a dispersive element. For example, by slightly detuning a resonant circuit from the signal $V(t)$ the frequency fluctuations $(1/2\pi)]\dot{\varphi}(t)$ are converted to amplitude fluctuations of the output signal. Provided the input amplitude fluctuations $[\epsilon(t)]/V_0$ are insignificant, the output amplitude fluctuations can be a good measure of the frequency fluctuations. Obviously, more sophisticated frequency discriminators exist (e.g., the cesium beam).

From the analog voltage one may use analog spectrum analyzers to determine $S_y(f)$, the frequency stability. By converting to digital data, other analyses are possible on a computer.

F. Common Hazards

1) Errors Caused by Signal-Processing Equipment: The intent of most frequency stability measurements is to evaluate the source and not the measuring equipment. Thus, one must know the performance of the measuring system. Of obvious importance are such aspects of the measuring equipment as noise level, dynamic range, resolution (dead time), and frequency range.

It has been pointed out that the noise bandwidth f_h is very essential for the mathematical convergence of certain expressions. Insofar as one wants to measure the signal *source*, one must know that the measuring system is not limiting the frequency response. At the very least, one must recognize that the frequency limit of the measuring system may be a very important, implicit parameter for either $\sigma_y^2(t)$ or $S_y(f)$. Indeed, one must account for any deviations of the measuring system form ideality such as a "nonflat" frequency response of the spectrum analyzer itself.

Almost any electronic circuit that processes a signal will, to some extent, convert amplitude fluctuations at the input terminals into phase fluctuations at the output. Thus, AM noise at the input will cause a time-varying phase (or FM noise) at the output. This can impose important constraints on limiters and automatic gain control (AGC) circuits when good frequency stability is needed. Similarly, this imposes constraints on equipment used for frequency stability measurements.

2) Analog Spectrum Analyzers (Frequency Domain): Typical analog spectrum analyzers are very similar in design to radio receivers of the superheterodyne type, and thus certain design features are quite similar. For example, image rejection (related to predetection bandwidth) is very important. Similarly, the actual shape of the analyzer's frequency window is important since this affects spectral resolution. As with receivers, dynamic range can be critical for the analysis of weak signals in the presence of substantial power in relatively narrow bandwidths (e.g., 60 Hz).

The slewing rate of the analyzer must be consistent with the analyzer's frequency window and the post-detection bandwidth. If one has a frequency window of 1 Hz, one cannot reliably estimate the intensity of a bright line unless the slewing rate is much slower than 1 Hz/s. Additional post-detection filtering will further reduce the maximum usable slewing rate.

3) Spectral Density Estimation from Time Domain Data: It is beyond the scope of this paper to present a comprehensive list of hazards for spectral density estimation; one should consult the literature [2]–[5]. There are a few points, however, which are worthy of special notice: a) data aliasing (similar to predetection bandwidth problems); b) spectral resolution; and c) confidence of the estimate.

4) Variances of Frequency Fluctuations $\sigma_y^2(\tau)$: It is not uncommon to have discrete frequency modulation of a source such as that associated with the power supply frequencies. The existence of discrete frequencies in $S_y(f)$ can cause $\sigma_y^2(\tau)$ to be a very rapidly changing function of τ. An interesting situation results when τ is an exact multiple of the period of the modulation frequency (e.g., one makes $\tau = 1$ s and there exists 60-Hz frequency modulation on the signal). In this situation, $\sigma_y^2(\tau = 1 \text{ s})$ can be very optimistic relative to values with slightly different values of τ.

One also must be concerned with the convergence properties of $\sigma_y^2(\tau)$ since not all noise processes will have finite limits to the estimates of $\sigma_y^2(\tau)$ (see Appendix I). One must be as critically aware of any "dead time" in the measurement process as of the system bandwidth.

5) Signal Source and Loading: In measuring frequency stability one should specify the exact location in the circuit from which the signal is obtained and the nature of the load used. It is obvious that the transfer characteristics of the device being specified will depend on the load and that the measured frequency stability might be affected. If the load itself is not constant during the measurements, one expects large effects on frequency stability.

6) Confidence of the Estimate: As with any measurement in science, one wants to know the confidence to assign to numerical results. Thus, when one measures $S_y(f)$ or $\sigma_y^2(\tau)$, it is important to know the accuracies of these estimates.

a) The Allan Variance: It is apparent that a single sample variance $\sigma_y^2(4, \tau, \tau)$ does not have good confidence, but, by averaging many independent samples, one can improve the accuracy of the estimate greatly. There is a key point in this statement, "independent samples." For this argument to be true, it is important that one sample variance be independent of the next. Since $\sigma_y^2(2, \tau, \tau)$ is related to the first difference of the frequency (11), it is sufficient that the noise perturbing $y(t)$ have "independent increments," i.e., that $y(t)$ be a random walk. In other words, it is sufficient that $S_y(f) \sim f^{-2}$ for low frequencies. One can show that for noise processes that are more divergent at low frequencies than f^{-2}, it is difficult (or impossible) to gain good confidence on estimates of $\sigma_y^2(\tau)$. For noise processes that are less divergent than f^{-2}, no problem exists.

It is worth noting that if we were interested in $\sigma_y^2(N = \infty, \tau, \tau)$, then the limit noise would become $S_y(f) \sim f^0$ instead of f^{-2} as it is for $\sigma_y^2(2, \tau, \tau)$. Since most real signal generators possess low-frequency divergent noises, $\langle\sigma_y^2(2, \tau, \tau)\rangle$ is more useful than $\sigma_y^2(N = \infty, \tau, \tau)$.

Although the sample variances $\sigma_y^2(2, \tau, \tau)$ will not be normally distributed, the variance of the average of m

independent (nonoverlapping) samples of $\sigma_y^2(2, \tau, \tau)$ (i.e., the variance of the Allan variance) will decrease as $1/m$ provided the conditions on low-frequency divergence are met. For sufficiently large m, the distribution of the m sample averages of $\sigma_y^2(2, \tau, \tau)$ will tend toward normal (central limit theorem). It is thus possible to estimate confidence intervals based on the normal distribution.

As always, one may be interested in τ values approaching the limits of available data. Clearly, when one is interested in τ values of the order of a year, one is severely limited in the size of m, the number of samples of $\sigma_y^2(2, \tau, \tau)$. Unfortunately, there seems to be no substitute for many samples and one extends τ at the expense of confidence in the results. "Truth in packaging" dictates that the sample size m be stated with the results.

b) Spectral Density: As before, one is referred to the literature for discussions of spectrum estimation [2]–[5]. It is worth pointing out, however, that for $S_y(f)$ there are basically two different types of averaging that can be employed: sample averaging of independent estimates of $S_y(f)$, and frequency averaging where the resolution bandwidth is made much greater than the reciprocal data length.

VIII. Conclusions

A good measure of frequency stability is the spectral density $S_y(f)$ of fractional frequency fluctuations $y(t)$. An alternative is the expected variance of N sample averages of $y(t)$ taken over a duration τ. With the beginning of successive sample periods spaced every T units of time, the variance is denoted by $\sigma_y^2(N, T, \tau)$. The stability measure, then, is the expected value of many measurements of $\sigma_y^2(N, T, \tau)$ with $N = 2$ and $T = \tau$; that is, $\sigma_y^2(\tau)$. For all real experiments one has a finite bandwidth. In general, the time domain measure of frequency stability $\sigma_y^2(\tau)$ is dependent on the noise bandwidth of the system. Thus, there are four important parameters to the time domain measure of frequency stability.

- N Number of sample averages ($N = 2$ for preferred measure).
- T Repetition time for successive sample averages ($T = \tau$ for preferred measure).
- τ Duration of each sample average.
- f_h System noise bandwidth.

Translations among the various stability measures for common noise types are possible, but there are significant reasons for choosing $N = 2$ and $T = \tau$ for the preferred measure of frequency stability in the time domain. This measure, the Allan variance, ($N = 2$) has been referenced by [12], [20]–[22] and more.

Although $S_y(f)$ appears to be a function of the single variable f, actual experimental estimation procedures for the spectral density involve a great many parameters. Indeed, its experimental estimation can be at least as involved as the estimation of $\sigma_y^2(\tau)$.

Appendix I

We want to derive (23) in the text. Starting from (10) we have

$$\begin{aligned}\langle\sigma_y^2(N, T, \tau)\rangle &= \left\langle \frac{1}{N-1}\sum_{n=1}^{N}\left(\bar{y}_n - \frac{1}{N}\sum_{k=1}^{N}\bar{y}_k\right)^2\right\rangle \\ &= \frac{1}{N-1}\left\{\sum_{n=1}^{N}\langle\bar{y}_n^2\rangle - \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\langle\bar{y}_i\bar{y}_j\rangle\right\} \\ &= \frac{1}{(N-1)\tau^2}\left\{\sum_{n=1}^{N}\int_{t_n}^{t_n+\tau}dt''\int_{t_n}^{t_n+\tau}dt'\,\langle y(t')y(t'')\rangle\right. \\ &\quad \left. - \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{t_j}^{t_j+\tau}dt''\int_{t_i}^{t_i+\tau}dt'\,\langle y(t')y(t'')\rangle\right\}\end{aligned} \tag{56}$$

where (9) has been used. Now

$$\langle y(t')y(t'')\rangle = R_y(t' - t'') \tag{57}$$

where $R_y(\tau)$ is the autocorrelation function of $y(t)$ and is the Fourier transform of $S_y(f)$, the power spectral density of $y(t)$. Equation (57) is true provided that $y(t)$ is stationary (at least in the wide or covariance sense), and that the average exists. If we assume the power spectral density of $y(t)$, $S_y(f)$ has low and high frequency cutoffs f_l and f_h (if necessary) so that

$$\int_0^\infty S_y(f)\,df$$

exists, then if y is a random variable, the average does exist and we may safely assume stationarity.

In practice, the high-frequency cutoff f_h is always present either in the device being measured or in the measuring equipment itself. When the high-frequency cutoff is necessary for convergence of integrals of $S_y(f)$ (or is too low in frequency), the stability measure will depend on f_h. The latter case can occur when the measuring equipment is too narrow-band. In fact, a useful type of spectral analysis may be done by varying f_h purposefully [18].

The low-frequency cutoff f_l may be taken to be much smaller than the reciprocal of the longest time of interest. The results of calculations as well as measurements will be meaningful if they are independent of f_l as f_l approaches zero. The range of exponents in power law spectral densities for which this is true will be discussed and are given in Fig. 1.

To continue, the derivation requires the Fourier transform relationships between the autocorrelation function and the power spectral density

$$\begin{aligned}S_y(f) &= 4\int_0^\infty R_y(\tau)\cos 2\pi f\tau\,d\tau \\ R_y(\tau) &= \int_0^\infty S_y(f)\cos 2\pi f\tau\,df.\end{aligned} \tag{58}$$

Using (58) and (57) in (56) gives

$$\begin{aligned}\langle\sigma_y^2(N,T,\tau)\rangle &= \frac{1}{(N-1)\tau^2}\Big\{\sum_{n=1}^{N}\int_0^\infty df\, S_y(f)\int_{t_n}^{t_n+\tau} dt'' \\ &\quad\cdot\int_{t_n}^{t_n+\tau} dt' \cos 2\pi f(t'-t'') - \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\int_0^\infty df\, S_y(f) \\ &\quad\cdot\int_{t_i}^{t_i+\tau} dt''\int_{t_j}^{t_j+\tau} dt' \cos 2\pi f(t'-t'')\Big\} \\ &= \frac{1}{(N-1)\tau^2}\Big\{\sum_{n=1}^{N}\Big[\int_0^\infty df\, S_y(f)\frac{\sin^2\pi f\tau}{(\pi f)^2}\Big] \\ &\quad - \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\Big[\int_0^\infty df\,\frac{S_y(f)}{(2\pi f)^2}\,(2\cos 2\pi fT(j-i) \\ &\quad - \cos 2\pi f[T(j-i)+\tau] - \cos 2\pi f[T(j-i)-\tau])\Big]\Big\}.\end{aligned} \tag{59}$$

(The interchanges in order of integration are permissible here since the integrals are uniformly convergent with the given restrictions on $S_y(f)$.) The first summation in the curly brackets is independent of the summation index n and thus gives just

$$N\int_0^\infty df\, S_y(f)\frac{\sin^2\pi f\tau}{(\pi f)^2}. \tag{60}$$

The kernel in the second term in the curly brackets may be further simplified

$$\begin{aligned}&2\cos 2\pi fT(j-i) - \cos 2\pi f(T(j-i)+\tau) \\ &\quad - \cos 2\pi f(T(j-i)-\tau) = 4\sin^2\pi f\tau\cos 2\pi fT(j-i).\end{aligned} \tag{61}$$

The second term is then

$$-\frac{1}{N}\Big(\int_0^\infty df\,\frac{S_y(f)}{(\pi f)^2}\sin^2\pi f\tau\sum_{i=1}^{N}\sum_{j=1}^{N}\cos 2\pi fT(j-i)\Big). \tag{62}$$

(The interchange of summation and integration is justified.) We must now do the double sum. Let

$$\begin{aligned} j - i &= k \\ 2\pi fT &= x. \end{aligned} \tag{63}$$

Changing summation indices from i and j to i and k gives for the sum

$$S = \sum_{i=1}^{N}\sum_{j=1}^{N}\cos x(j-i) = \sum_{i=1}^{N}\sum_{k=1-i}^{N-i}\cos kx. \tag{64}$$

The region of summation over the discrete variables i and k is shown in Fig. 5 for $N = 4$.

The summand is independent of i so that one may interchange the order of summation and sum over i first. The summand is even in k and the contributions for $k < 0$ are equal to those for $k > 0$, and so we may pull out the term for $k = 0$ separately and write

$$\begin{aligned} S &= 2\Big(\sum_{k=1}^{N-1}\cos kx\sum_{i=1}^{N-k}1\Big) + \sum_{i=1}^{N}1 \\ &= 2\Big(\sum_{k=1}^{N-1}(N-k)\cos kx\Big) + N. \end{aligned} \tag{65}$$

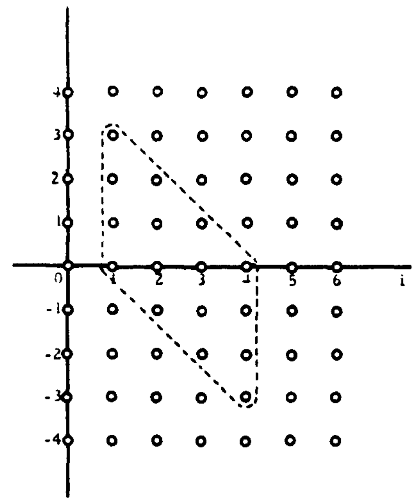

Fig. 5. Region of summation for i and k for $N = 4$.

This may be written as

$$S = N + 2\,\mathrm{Re}\Big[N - \frac{1}{i}\frac{d}{dx}\Big]\sum_{k=1}^{N-1} e^{ikx} \tag{66}$$

where $\mathrm{Re}[U]$ means the real part of U and d/dx is the differential operator. The series is a simple geometric series and may be summed easily, giving

$$\begin{aligned} S &= N + 2\,\mathrm{Re}\Big\{\Big[N - \frac{1}{i}\frac{d}{dx}\Big]\frac{e^{ix} - e^{iNx}}{1 - e^{ix}}\Big\} \\ &= N + 2\,\mathrm{Re}\Big\{\frac{1 - e^{iNx} - N(1 - e^{ix})}{4\sin^2 x/2}\Big\} \\ &= \frac{\sin^2 Nx/2}{\sin^2 x/2}. \end{aligned} \tag{67}$$

Combining everything we get, after some rearrangement,

$$\begin{aligned}&\langle\sigma_y^2(N,T,\tau)\rangle \\ &\quad = \frac{N}{N-1}\int_0^\infty df\, S_y(f)\frac{\sin^2\pi f\tau}{(\pi f\tau)^2}\Big[1 - \frac{\sin^2\pi r fN\tau}{N^2\sin^2\pi r f\tau}\Big]\end{aligned} \tag{68}$$

where $r = T/\tau$. This is the result given in (23).

We can determine a number of things very easily from this equation. First let us change variables. Let $\pi f\tau = u$, then

$$\begin{aligned}&\langle\sigma_y^2(N,T,\tau)\rangle \\ &\quad = \frac{N}{(N-1)\pi\tau}\int_0^\infty du\, S_y\Big(\frac{u}{\pi\tau}\Big)\frac{\sin^2 u}{u^2}\Big\{1 - \frac{\sin^2 Nru}{N^2\sin^2 ru}\Big\}.\end{aligned} \tag{69}$$

The kernel behaves like u^2 as $u \to 0$ and like u^{-2} as $u \to \infty$. Therefore $\langle\sigma_y^2(N, T, \tau)\rangle$ is convergent for power law spectral densities, $S_y(f) = h_\alpha f^\alpha$, without any low- or high-frequency cutoffs for $-3 < \alpha < 1$. Using (69) for power law spectral densities we find

$$\begin{aligned}\langle\sigma_y^2(N,T,\tau)\rangle &= \tau^{-\alpha-1}h_\alpha C_\alpha, \qquad -3 < \alpha < 1 \\ &= \tau^{\mu}h_\alpha C_\alpha, \qquad \mu \equiv -\alpha - 1\end{aligned}$$

and

$$C_\alpha \equiv \frac{N}{(N-1)\pi^{\alpha+1}} \int_0^\infty du\, u^\alpha \frac{\sin^2 u}{u^2}\left\{1 - \frac{\sin^2 Nru}{N^2 \sin^2 ru}\right\}. \tag{70}$$

This is the basis for the plot in Fig. 1 in the text of μ versus α. For $\alpha \geq 1$ we must include the high-frequency cutoff f_h.

For $N = 2$ and $r = 1$ the results are particularly simple. We have

$$\langle \sigma_y^2(2, \tau, \tau)\rangle = \tau^{-\alpha-1} h_\alpha \frac{2}{\pi^{\alpha+1}} \int_0^\infty du\, u^{\alpha-2} \sin^4 u \tag{71}$$

for power law spectral densities. For $N = 2$ and general r we get

$$\langle \sigma_y^2(2, T, \tau)\rangle$$

$$= \frac{1}{2\pi\tau}\int_0^\infty du\, S_y\left(\frac{u}{\pi\tau}\right) \cdot \frac{1 - \cos 2u - \cos 2ru + \frac{\cos 2u(r+1)}{2} + \frac{\cos 2u(r-1)}{2}}{u^2}$$

$$= \frac{2}{\pi\tau}\int_0^\infty du\, S_y\left(\frac{u}{\pi\tau}\right)\frac{\sin^2 u \sin^2 ru}{u^2}. \tag{72}$$

The first form in (72) is particularly simple and is also useful for $r = 1$ in place of (71).

Let us discuss the case for $\alpha \geq 1$ in a little more detail. As mentioned above we must include the high-frequency cutoff f_h for convergence. The general behavior can be seen most easily from (68). After placing the factor τ^{-2} outside the integral and combining the factor f^{-2} with $S_y(f)$ we find that the remaining part of the kernel consists of some constants and some oscillatory terms. If $2\pi f_h\tau \gg 1$ it is apparent that the rapidly oscillating terms contribute very little to the integral. Most of the contribution comes from the integral over the constant term causing the major portion of the τ dependence to be the τ^{-2} factor outside the integral. This is the reason for the vertical slope at $\mu = -2$ in the μ versus α plot in Fig. 1 in the text.

One other point deserves some mention. The constant term of the kernel discussed in the preceding paragraph is different for $r = 1$ from the value for $r \neq 1$. This is readily seen from (72) for $N = 2$; for $r = 1$ the constant term is 3/2 while for $r \neq 1$ it is 1. This is the reason for $\delta_k(r-1)$, which appears in some of the results of Appendix II. In practice, $\delta_k(r-1)$ does not have zero width but is smeared out over a width of approximately $(2\pi f_h\tau)^{-1}$. If there must be dead time $r \neq 1$, it is wise to choose $(r-1) \gg (2\pi f_h\tau)^{-1}$ or $(r-1) \ll (2\pi f_h\tau)^{-1}$ but with $2\pi f_h\tau \gg 1$. In the latter case, one may assume $r \approx 1$.

Appendix II

Let $y(t)$ be a sample function of a random noise process with a spectral density $S_y(f)$. The function $y(t)$ is assumed to be pure real and $S_y(f)$ is a one-sided spectral density relative to a cycle frequency (i.e., the dimensions of $S_y(f)$ are that of y^2 per hertz). (For additional information see Appendix I, [7], [8], [18].)

Let $x(t)$ be defined by the equation

$$\dot{x}(t) \equiv \frac{dx}{dt} \equiv y(t). \tag{73}$$

Define the following. t_0 is arbitrary instant of time and

$$t_{n+1} \equiv t_n + T, \qquad n = 0, 1, 2, \cdots, \tag{74}$$

$$\bar{y}_n \equiv \frac{1}{\tau}\int_{t_n}^{t_n+\tau} y(t)\, dt = \frac{x(t_n+\tau) - x(t_n)}{\tau} \tag{75}$$

$$\langle \bar{y}\rangle_N \equiv \frac{1}{N}\sum_{n=1}^N \bar{y}_n, \tag{76}$$

and let f_h be a high-frequency cutoff (infinitely sharp) with $2\pi f_h\tau \gg 1$.

Definition:

$$\langle \sigma_y^2(N, T, \tau)\rangle \equiv \left\langle \frac{1}{N-1}\sum_{n=1}^N (\bar{y}_n - \langle \bar{y}\rangle_N)^2\right\rangle. \tag{77}$$

Special Case:

$$\langle \sigma_y^2(2, T, \tau)\rangle = \left\langle \frac{(\bar{y}_2 - \bar{y}_1)^2}{2}\right\rangle. \tag{78}$$

Special Case:

$$\sigma_y^2(\tau) \equiv \langle \sigma_y^2(2, \tau, \tau)\rangle = \left\langle \frac{[x(t_0 + 2\tau) - 2x(t_0+\tau) + x(t_0)]^2}{2\tau^2}\right\rangle. \tag{79}$$

Definition:

$$D_x^2(\tau) \equiv \langle [x(t_0 + 2\tau) - 2x(t_0+\tau) + x(t_0)]^2\rangle. \tag{80}$$

Consequence of Definitions:

$$D_x^2(\tau) = 2\tau^2\sigma_y^2(\tau) \equiv 2\sigma_x^2(\tau). \tag{81}$$

Definition:

$$\psi_x^2(T, \tau) \equiv \langle [x(t_0 + T + \tau) - x(t_0 + T) - x(t_0+\tau) + x(t_0)]^2\rangle. \tag{82}$$

Consequence of Definitions:

$$\psi_x^2(T, \tau) = 2\tau^2\langle \sigma_y^2(2, T, \tau)\rangle. \tag{83}$$

Special Case:

$$\psi_x^2(\tau, \tau) = D_x^2(\tau). \tag{84}$$

Random Walk y

$$S_y(f) = \frac{h_{-2}}{f^2} \qquad \left(S_x(f) = \frac{h_{-2}}{(2\pi)^2 f^4}\right)$$

$$r \equiv \frac{T}{\tau}, \qquad 0 \leq f \leq f_h.$$

Quantity	*Relation*	
$\langle \sigma_y^2(N, T, \tau)\rangle$	$h_{-2}\cdot\frac{(2\pi)^2\,\lvert\tau\rvert}{12}\,[r(N+1)-1], \quad r \geq 1$	(85)
$\langle \sigma_y^2(N, \tau, \tau)\rangle$	$h_{-2}\cdot\frac{(2\pi)^2\,\lvert\tau\rvert}{12}\cdot N, \quad r = 1$	(86)
$\sigma_y^2(\tau)$	$h_{-2}\cdot\frac{(2\pi)^2\,\lvert\tau\rvert}{6}, \quad N = 2, r = 1$	(87)
$D_x^2(\tau) = 2\sigma_x^2(\tau)$	$h_{-2}\cdot\frac{2(2\pi)^2\,\lvert\tau\rvert^3}{6}$	(88)
$\psi_x^2(T, \tau)$	$h_{-2}\cdot\frac{(2\pi)^2\,\lvert\tau\rvert^3}{6}(3r-1), \quad r \geq 1$ $h_{-2}\cdot\frac{(2\pi)^2\,\lvert T\rvert^3}{6}\left(\frac{3}{r}-1\right), \quad r \leq 1\,.$	(89)

Flicker y

$$S_y(f) = \frac{h_{-1}}{f} \qquad \left(S_x(f) = \frac{h_{-1}}{(2\pi)^2 f^3}\right)$$

$$r = T/\tau, \qquad 0 \leq f \leq f_h\,.$$

Quantity	*Relation*	
$\langle \sigma_y^2(N, T, \tau)\rangle$	$h_{-1}\cdot\frac{1}{N(N-1)}\sum_{n=1}^{N}(N-n)\cdot[-2(nr)^2\ln(nr) + (nr+1)^2\ln(nr+1) + (nr-1)^2\ln\lvert nr-1\rvert]$	(90)
$\langle \sigma_y^2(N, \tau, \tau)\rangle$	$h_{-1}\cdot\frac{N\ln N}{N-1}, \quad (r = 1)$	(91)
$\sigma_y^2(\tau)$	$h_{-1}\cdot 2\ln 2, \; (N = 2, r = 1)$	(92)
$D_x^2(\tau) = 2\sigma_x^2(\tau)$	$h_{-1}\cdot 4\tau^2\ln 2$	(93)
$\psi_x^2(T, \tau)$	$h_{-1}\cdot\tau^2[-2r^2\ln r + (r+1)^2\ln(r+1) + (r-1)^2\ln\lvert r-1\rvert]$	(94)
	$\sim h_{-1}\cdot 2\tau^2(2+\ln r), \quad r \gg 1$ $\sim h_{-1}\cdot 2T^2(2-\ln r), \quad r \ll 1\,.$	(95) *

White y (Random Walk x)

$$S_y(f) = h_0 \qquad S_x(f) = \frac{h_0}{(2\pi)^2 f^2}$$

$$r = T/\tau, \qquad 0 \leq f \leq f_h\,.$$

Quantity	*Relation*	
$\langle \sigma_y^2(N, T, \tau)\rangle$	$\frac{h_0}{2}\cdot\lvert\tau\rvert^{-1}, \quad r \geq 1$ $h_0\cdot\frac{1}{6}r(N+1)\,\lvert\tau\rvert^{-1}, \quad Nr \leq 1$	(96)
$\langle \sigma_y^2(N, \tau, \tau)\rangle$	$\frac{h_0}{2}\cdot\lvert\tau\rvert^{-1}, \quad r = 1$	(97)
$\sigma_y^2(\tau)$	$\frac{h_0}{2}\cdot\lvert\tau\rvert^{-1}, \quad N = 2, r = 1$	(98)
$D_x^2(\tau) = 2\sigma_x^2(\tau)$	$h_0\cdot\lvert\tau\rvert$	(99)
$\psi_x^2(T, \tau)$	$h_0\cdot\lvert\tau\rvert, \quad r \geq 1$ $h_0\cdot T, \quad r \leq 1\,.$	(100)

* See Appendix Note # 21

Flicker x

$$S_y(f) = h_1\,\lvert f\rvert\left(S_x(f) = \frac{h_1}{(2\pi)^2 f}\right)$$

$$r = T/\tau, \qquad 2\pi f_h\tau \gg 1, \qquad 2\pi f_h T \gg 1, \qquad 0 \leq f \leq f_h.$$

Quantity	*Relation*	
$\langle \sigma_y^2(N, T, \tau)\rangle$	$h_1\cdot\frac{2}{(2\pi\tau)^2}\left\{2+\ln(2\pi f_h\tau) + \frac{1}{N(N-1)}\sum_{n=1}^{N-1}(N-n)\cdot\ln\left[\frac{n^2r^2}{n^2r^2-1}\right]\right\}, \quad r \gg 1$	(101) **
$\langle \sigma_y^2(N, \tau, \tau)\rangle$	$h_1\cdot\frac{2(N+1)}{N\tau^2(2\pi)^2}\left[2+\ln(2\pi f_h\tau) - \frac{\ln N}{N^2-1}\right], \quad r = 1$	(102) **
$\sigma_y^2(\tau)$	$h_1\cdot\frac{1}{\tau^2(2\pi)^2}\{3[2+\ln(2\pi f_h\tau)] - \ln 2\}, \quad N = 2, r = 1$	(103) **
$D_x^2(\tau) = 2\sigma_x^2(\tau)$	$\frac{h_1}{(2\pi)^2}\cdot 2\{3[2+\ln(2\pi f_h\tau)] - \ln 2\}$	(104) **
$\psi_x^2(T, \tau)$	$h_1\cdot\frac{4}{(2\pi)^2}[2+\ln(2\pi f_h\tau)], \quad r \gg 1$ $h_1\cdot\frac{2}{(2\pi)^2}\{3[2+\ln(2\pi f_h\tau)] - \ln 2\}, \quad r = 1$	(105) **
	$h_1\cdot\frac{4}{(2\pi)^2}[2+\ln(2\pi f_h T)], \quad r \ll 1\,.$	

White x

$$S_y(f) = h_2 f^2 \qquad \left(S_x(f) = \frac{h_2}{(2\pi)^2}\right)$$

$$r = T/\tau;\; \delta_k(r-1) = \begin{cases}1, & \text{if } r = 1\\ 0, & \text{otherwise}\end{cases}$$

$$2\pi f_h\tau \gg 1, \qquad 0 \leq f \leq f_h\,.$$

Quantity	*Relation*	
$\langle \sigma_y^2(N, T, \tau)\rangle$	$h_2\cdot\frac{N+\delta_k(r-1)}{N(2\pi)^2}\cdot\frac{2f_h}{\tau^2}$	(106)
$\langle \sigma_y^2(N, \tau, \tau)\rangle$	$h_2\cdot\frac{N+1}{N(2\pi)^2}\cdot\frac{2f_h}{\tau^2}, \quad r = 1$	(107)
$\sigma_y^2(\tau)$	$h_2\cdot\frac{3f_h}{(2\pi)^2\tau^2}, \quad N = 2, r = 1$	(108)
$D_x^2(\tau) = 2\sigma_x^2(\tau)$	$h_2\cdot\frac{6f_h}{(2\pi)^2}$	(109)
$\psi_x^2(T, \tau)$	$h_2\cdot[2+\delta_k(r-1)]\,\frac{2f_h}{(2\pi)^2}$	(110)

** See Appendix Note # 22

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References

[1] E. T. Jaynes, "Information theory and statistical mechanics," *Phys. Rev.*, vol. 108, sec. 15, Oct. 1957, pp. 171–190.

[2] C. Bingham, M. D. Godfrey, and J. W. Tukey, "Modern techniques of power spectrum estimation," *IEEE Trans. Audio Electrocoust.*, vol. AU-15, June 1967, pp. 56–66.

[3] R. B. Blackman, *Linear Data Smoothing and Prediction in Theory and Practice*. Reading, Mass.: Addison-Wesley, 1965.

[4] R. B. Blackman and J. W. Tukey, *The Measurement of Power Spectra*. New York: Dover, 1958.

[5] E. O. Brigham and R. E. Morrow, "The fast Fourier transform," *IEEE Spectrum*, vol. 4, Dec. 1967, pp. 63–70.

[6] E. J. Baghdady, R. D. Lincoln, and B. D. Nelin. "Short-term frequency stability: theory, measurement, and status," *Proc. IEEE-NASA Symp. on Short-Term Frequency Stability* (NASA SP-80), Nov. 1964, pp. 65–87; also in *Proc. IEEE*, vol. 53, July 1965, pp. 704–722 and *Proc. IEEE* (Corresp.), vol 53, Dec. 1965. pp. 2110–2111.

[7] L. Cutler and C. Searle, "Some aspects of the theory and measurement of frequency fluctuations in frequency standards." *Proc. IEEE*, vol. 54, Feb. 1966, pp. 136–154.

[8] D. W. Allan, "Statistics of atomic frequency standards," *Proc. IEEE*, vol. 54, Feb. 1966, pp. 221–230.

[9] N. A. Shewart, *Economic Control of Quality of Manufactured Product*. Princeton, N. J.: Van Nostrand, 1931, p. 146.

[10] J. A. Barnes, "Atomic timekeeping and the statistics of precision signal generators," *Proc. IEEE*, vol. 54, Feb. 1966, pp. 207–220.

[11] J. A. Barnes and D. W. Allan, "An approach to the prediction of coordinated universal time," *Frequency*, vol. 5, Nov.–Dec. 1967, pp. 15–20.

[12] R. F. C. Vessot *et al.*, "An intercomparison of hydrogen and cesium frequency standards," *IEEE Trans. Instrum. Meas.*, vol. IM-15, Dec. 1966, pp. 165–176.

[13] J. A. Barnes, "Tables of bias functions, B_1 and B_2, for variances based on finite samples of processes with power law spectral densities," NBS, Washington, D. C., Tech. Note 375, Jan. 1969.

[14] D. B. Leeson and G. F. Johnson, "Short-term stability for a Doppler radar: requirements, measurements, and techniques," *Proc. IEEE*, vol. 54, Feb. 1966, pp. 244–248.

[15] W. K. Saunders in *Radar Handbook*, M. I. Skolnik, Ed. New York: McGraw Hill, 1970, ch. 16.

[16] R. S. Raven, "Requirements on master oscillators for coherent radar," *Proc. IEEE*, vol. 54, Feb. 1966, pp. 237–243.

[17] D. B. Leeson, "A simple model of feedback oscillator noise spectrum," *Proc. IEEE* (Lett.), vol. 54, Feb. 1966, pp. 329–330.

[18] R. F. C. Vessot, L. Mueller, and J. Vanier, "The specification of oscillator characteristics from measurements made in the frequency domain," *Proc. IEEE*, vol. 54, Feb. 1966. pp. 199–207.

[19] Floyd M. Gardner, *Phaselock Techniques*. New York: Wiley, 1966.

[20] C. Menoud, J. Racine, and P. Kartaschoff, "Atomic hydrogen maser work at L.S.R.H., Neuchatel, Switzerland," *Proc. 21st Ann. Symp. Frequency Control*, Apr. 1967, pp. 543–567.

[21] A. G. Mungall, D. Morris, H. Daams, and R. Bailey, "Atomic hydrogen maser development at the National Research Council of Canada," *Metrologia*, vol. 4, July 1968, pp. 87–94.

[22] R. F. C. Vessot, "Atomic hydrogen masers, an introduction and progress report," *Hewlett-Packard J.*, vol. 20, Oct. 1968, pp. 15–20.

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