CSE 102
Introduction to Analysis of Algorithms
Asymptotic Growth of Functions (CLRS 3.1)

We introduce several types of asymptotic notation which are used to compare the relative performance and efficiency of algorithms. As we shall see, the asymptotic run time of an algorithm gives a simple machine independent characterization of its complexity.

Definition 1 Let $g(n)$ be a function. The set $O(g(n))$ is defined as

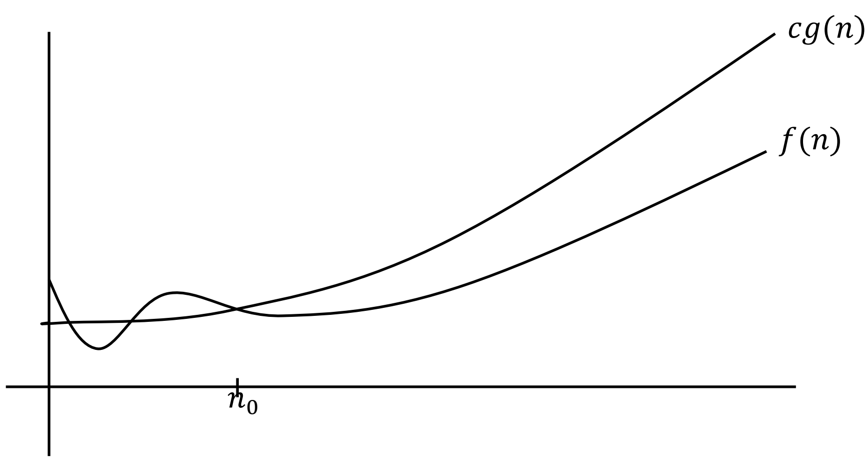
$$O(g(n)) = \{\, f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \geq n_0 \colon 0 \leq f(n) \leq cg(n) \,\}.$$

In other words, $f(n) \in O(g(n))$ if and only if there exist positive constants c, and n_0, such that for all $n \geq n_0$, the inequality $0 \leq f(n) \leq cg(n)$ is satisfied. We shall say in this case that $f(n)$ is *Big O* of $g(n)$, or that $g(n)$ is an *asymptotic upper bound* for $f(n)$.

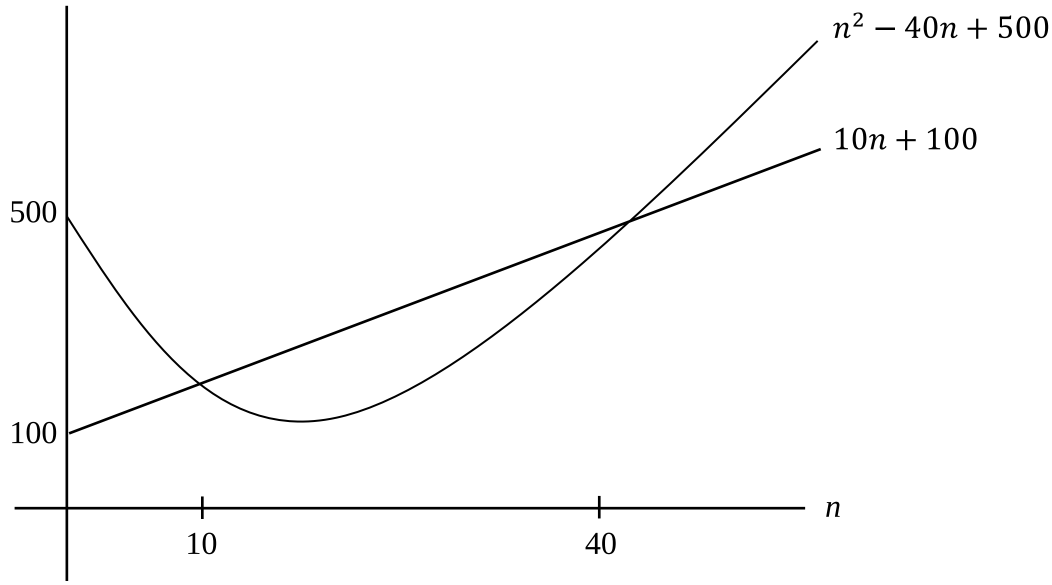
We often abuse notation slightly by writing $f(n) = O(g(n))$ to mean $f(n) \in O(g(n))$. Actually $f(n) \in O(g(n))$ is itself an abuse of notation. We should really write $f \in O(g)$ since what we have defined is a set of functions, not a set of numbers. The notational convention $O(g(n))$ is useful since it allows us to refer to the set $O(n^3)$ say, without having to introduce a new function symbol for the polynomial n^3. Observe that if $f(n) = O(g(n))$ then $f(n)$ is *asymptotically non-negative*, i.e. $f(n)$ is non-negative for all sufficiently large n, and likewise for $g(n)$. We make the blanket assumption from now on that all functions under discussion are asymptotically non-negative.

In practice we will be concerned with integer valued functions of a (positive) integer n ($g \colon \boldsymbol{Z}^+ \to \boldsymbol{Z}^+$). However, in what follows, it is useful to consider n to be a continuous real variable taking positive values and g to be real valued function ($g \colon \boldsymbol{R}^+ \to \boldsymbol{R}^+$).

Geometrically $f(n) = O(g(n))$ says:



Example 1 $10n + 100 = O(n^2 - 40n + 500)$. Observe $0 \le 10n + 100 \le n^2 - 40n + 500$ for all $n \ge 40$. Thus we may take $n_0 = 40$ and $c = 1$ in the definition.

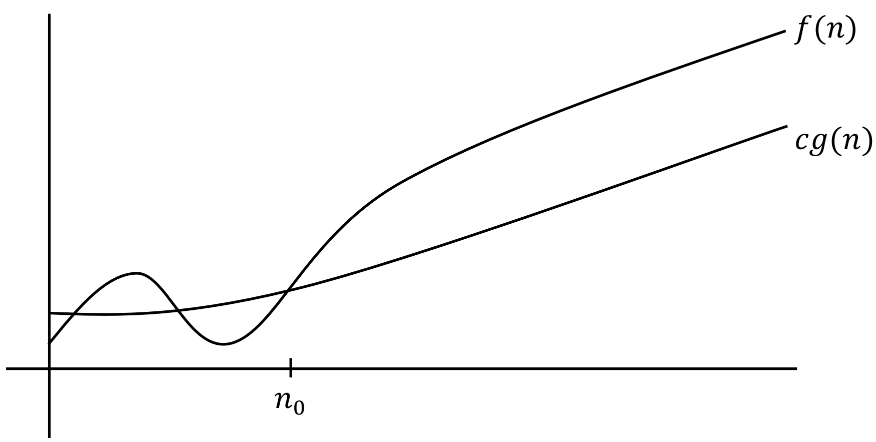


In fact $an + b = O(cn^2 + dn + e)$ for any constants *a*-*e*, and more generally $p(n) = O(q(n))$ whenever $p(n)$ and $q(n)$ are polynomials satisfying $\deg(p) \le \deg(q)$. (See exercises (c) and (d) at the end of this handout.)

Definition 2 Let $g(n)$ be a function and define the set $\Omega(g(n))$ to be

$$\Omega(g(n)) = \{\, f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \ge n_0 : \; 0 \le cg(n) \le f(n) \,\}.$$

We say $f(n)$ is *big Omega* of $g(n)$, and that $g(n)$ is an *asymptotic lower bound* for $f(n)$. As before we write $f(n) = \Omega(g(n))$ to mean $f(n) \in \Omega(g(n))$. The geometric interpretation is:



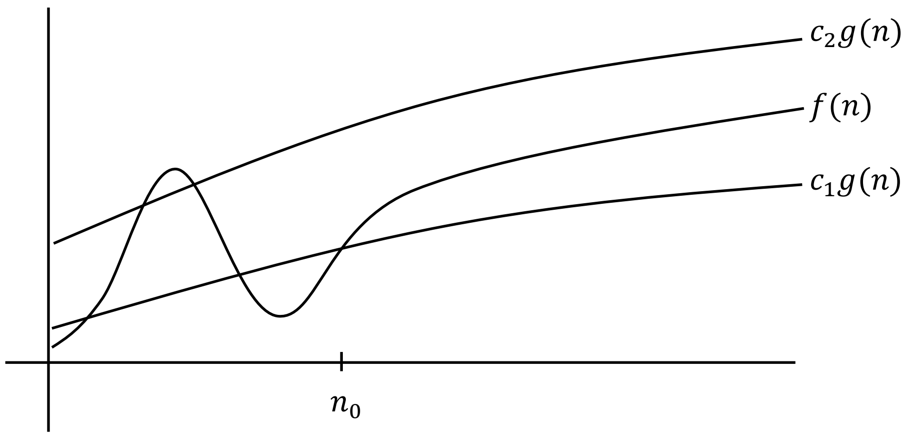
Theorem 1 $f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$.

Proof: If $f(n) = O\big(g(n)\big)$ then there exist positive numbers c_1, n_1 such that $0 \le f(n) \le c_1 g(n)$ for all $n \ge n_1$. Let $c_2 = \frac{1}{c_1}$ and $n_2 = n_1$. Then $0 \le c_2 f(n) \le g(n)$ for all $n \ge n_2$, proving $g(n) = \Omega\big(f(n)\big)$. The converse is similar and we leave it to the reader. ■

Definition 3 Let $g(n)$ be a function and define the set $\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$. Equivalently

$$\Theta(g(n)) = \{\, f(n) \mid \exists c_1 > 0, \exists c_2 > 0, \exists n_0 > 0, \forall n \ge n_0\colon 0 \le c_1 g(n) \le f(n) \le c_2 g(n) \,\}$$

We write $f(n) = \Theta(g(n))$ and say that $g(n)$ is an *asymptotically tight bound* for $f(n)$. We interpret this geometrically as:



Exercise 1 Prove that $f(n) = \Theta(g(n))$ if and only if $g(n) = \Theta(f(n))$.

Exercise 2 Let $g(n)$ be any function, and $c > 0$. Prove that $cg(n) = O\big(g(n)\big)$, and $cg(n) = \Omega\big(g(n)\big)$, whence $cg(n) = \Theta\big(g(n)\big)$.

Example 2 Prove that $\sqrt{n+10} = \Theta\big(\sqrt{n}\big)$.

Proof: According to the definition, we must find positive numbers c_1, c_2, n_0, such that the inequality $0 \le c_1\sqrt{n} \le \sqrt{n+10} \le c_2\sqrt{n}$ holds for all $n \ge n_0$. Pick $c_1 = 1$, $c_2 = \sqrt{2}$, and $n_0 = 10$. Then if $n \ge n_0$ we have:

$$
\begin{aligned}
& -10 \le 0 \quad \text{and} \quad 10 \le n \\
\therefore \quad & -10 \le (1-1)n \quad \text{and} \quad 10 \le (2-1)n \\
\therefore \quad & -10 \le (1-c_1^2)n \quad \text{and} \quad 10 \le (c_2^2-1)n \\
\therefore \quad & c_1^2 n \le n+10 \quad \text{and} \quad n+10 \le c_2^2 n, \\
\therefore \quad & c_1^2 n \le n+10 \le c_2^2 n, \\
\therefore \quad & c_1\sqrt{n} \le \sqrt{n+10} \le c_2\sqrt{n},
\end{aligned}
$$

as required. ■

The reader may find our choice of values for the constants c_1, c_2, n_0 somewhat mysterious. Adequate values for these constants can usually be obtained by working backwards algebraically from the inequality to be proved. Notice that in this example there are many valid choices. For instance one checks easily that $c_1 = \sqrt{1/2}$, $c_2 = \sqrt{3/2}$, and $n_0 = 20$ work equally well.

Exercise 3 Let a, b be real numbers, with $b > 0$. Prove directly from the definition that $(n+a)^b = \Theta(n^b)$. (By the end of this handout, we will learn a much easier way to prove this.)

Theorem 2 If $h(n) = O(g(n))$ and if $f(n) \le h(n)$ for all sufficiently large n, then $f(n) = O(g(n))$.

Proof: The above hypotheses say that there exist positive numbers c and n_1 such that $h(n) \le cg(n)$ for all $n \ge n_1$, and that there exists positive n_2 such that $0 \le f(n) \le h(n)$ for all $n \ge n_2$. (Recall all functions under discussion, in particular $f(n)$, are assumed to be asymptotically non-negative.) Then for all $n \ge n_0 = \max(n_1, n_2)$ we have $0 \le f(n) \le cg(n)$, showing that $f(n) = O(g(n))$. ■

Exercise 4 Prove that if $h_1(n) \le f(n) \le h_2(n)$ for all sufficiently large n, where $h_1(n) = \Omega(g(n))$ and $h_2(n) = O(g(n))$, then $f(n) = \Theta(g(n))$.

Example 3 Let $\boldsymbol{k \ge 1}$ be a fixed integer. Prove that $\boldsymbol{\sum_{i=1}^{n} i^k = \Theta(n^{k+1})}$.

Proof: Observe that $\sum_{i=1}^{n} i^k \le \sum_{i=1}^{n} n^k = n \cdot n^k = n^{k+1} = O(n^{k+1})$. Also

$$\begin{aligned}
\sum_{i=1}^{n} i^k &\ge \sum_{i=\lceil n/2 \rceil}^{n} i^k \\
&\ge \sum_{i=\lceil n/2 \rceil}^{n} \lceil n/2 \rceil^k \\
&= (n - \lceil n/2 \rceil + 1) \cdot \lceil n/2 \rceil^k \\
&= (\lfloor n/2 \rfloor + 1) \cdot \lceil n/2 \rceil^k \\
&> (n/2 - 1 + 1) \cdot (n/2)^k \\
&= (1/2)^{k+1} n^{k+1} \\
&= \Omega(n^{k+1})
\end{aligned}$$

By the preceding exercise we conclude $\sum_{i=1}^{n} i^k = \Theta(n^{k+1})$. ■

When asymptotic notation appears in a formula such as $f(n) = 3n^2 + \Theta(n)$, we interpret $\Theta(n)$ to stand for some function in the class $\Theta(n)$ which we do not care to specify. The expression $\sum_{i=1}^{n} \Theta(i)$ can be puzzling. Note that $\Theta(1) + \Theta(2) + \Theta(3) + \cdots + \Theta(n)$ is meaningless, since $\Theta(\text{constant})$ consists of all functions that are bounded above and below by constants. We interpret $\Theta(i)$ in this expression to stand for a *single* function $f(i)$ in the class $\Theta(i)$, evaluated at $i = 1, 2, \ldots, n$.

Exercise 5 Prove that $\sum_{i=1}^{n} \Theta(i) = \Theta(n^2)$. The left hand side stands for a single function $f(i)$ in $\Theta(i)$ summed over $i = 1, 2, 3, \ldots, n$. By the previous exercise it is sufficient to show that $h_1(n) \le \sum_{i=1}^{n} f(i) \le h_2(n)$ for all sufficiently large n, where $h_1(n) = \Omega(n^2)$ and $h_2(n) = O(n^2)$.

Definition 4 $o(g(n)) = \{ f(n) \mid \forall c > 0, \exists n_0 > 0, \forall n \ge n_0\colon\ 0 \le f(n) < cg(n) \}$. We say that $g(n)$ is a *strict Asymptotic upper bound* for $f(n)$ and write $f(n) = o(g(n))$ as before.

Lemma 1 $f(n) = o(g(n))$ if and only if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.

Proof: Observe that $f(n) = o(g(n))$ if and only if $\forall c > 0, \exists n_0 > 0, \forall n \geq n_0$: $0 \leq \frac{f(n)}{g(n)} < c$, which is the very definition of the limit statement $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$. ■

Example 4 $\ln(n) = o(n)$ since $\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$. (Hint: use l'Hopitals rule.)

Example 5 $n^k = o(e^n)$ for any $k > 0$ since $\lim_{n\to\infty} \frac{n^k}{e^n} = 0$. (Apply l'Hopitals rule $\lceil k \rceil$ times.) One shows similarly that $n^k = o(b^n)$ for any $b > 1$. In other words, any polynomial grows strictly slower than any exponential.

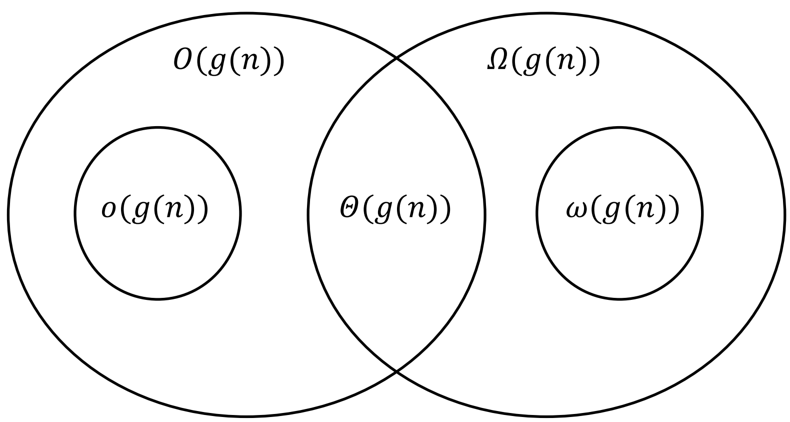
By comparing definitions of $o(g(n))$ and $O(g(n))$ one sees immediately that $o(g(n)) \subseteq O(g(n))$.

Exercise 6 Prove that $o(g(n)) \cap \Omega(g(n)) = \emptyset$ by verifying that no function can belong to both $o(g(n))$ and $\Omega(g(n))$. Thus $o(g(n)) \subseteq O(g(n)) - \Theta(g(n))$.

Definition 5 $\omega(g(n)) = \{ f(n) \mid \forall c > 0, \exists n_0 > 0, \forall n \geq n_0\text{: } 0 \leq cg(n) < f(n) \}$. Here we say that $g(n)$ is a *strict asymptotic lower bound* for $f(n)$ and write $f(n) = \omega(g(n))$.

Exercise 7 Prove $f(n) = \omega(g(n))$ if and only if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$. Also prove $\omega(g(n)) \cap O(g(n)) = \emptyset$, whence $\omega(g(n)) \subseteq \Omega(g(n)) - \Theta(g(n))$.

The following picture emerges.



Theorem 3 If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = L$, where $0 \leq L < \infty$, then $f(n) = O(g(n))$. (Note: the converse is false.)

Proof: The limit statement says $\forall \varepsilon > 0, \exists n_0 > 0, \forall n \geq n_0$: $\left|\frac{f(n)}{g(n)} - L\right| < \varepsilon$. Since this holds for all ε, we may set $\varepsilon = 1$. Then there exists a positive n_0 such that for all $n \geq n_0$:

$$\left|\frac{f(n)}{g(n)} - L\right| < 1$$

$$\therefore \quad -1 < \frac{f(n)}{g(n)} - L < 1$$

$$\therefore \quad \frac{f(n)}{g(n)} < L + 1$$

$$\therefore \quad f(n) < (L+1) \cdot g(n).$$

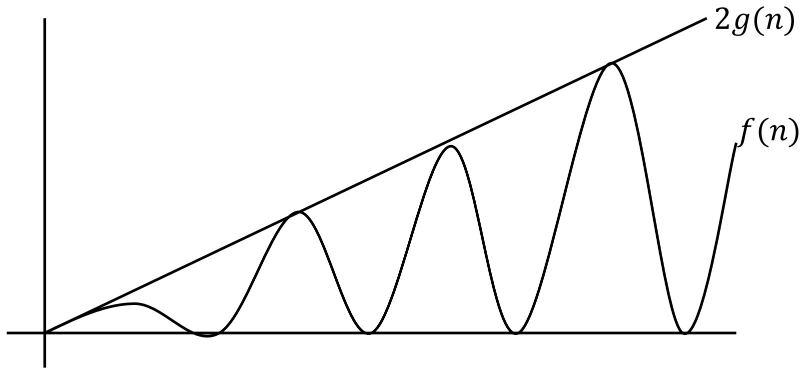
Now taking $c = L + 1$ in the definition of O yields $f(n) = O(g(n))$ as claimed. ■

Theorem 4 If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = L$, where $0 < L \le \infty$, then $f(n) = \Omega(g(n))$.
Proof: The limit statement implies $\lim_{n\to\infty} \frac{g(n)}{f(n)} = L'$, where $L' = 1/L$ and hence $0 \le L' < \infty$. By the preceding theorem $g(n) = O(f(n))$, and therefore $f(n) = \Omega(g(n))$. ■

Exercise 8 Prove that if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = L$, where $0 < L < \infty$, then $f(n) = \Theta(g(n))$.

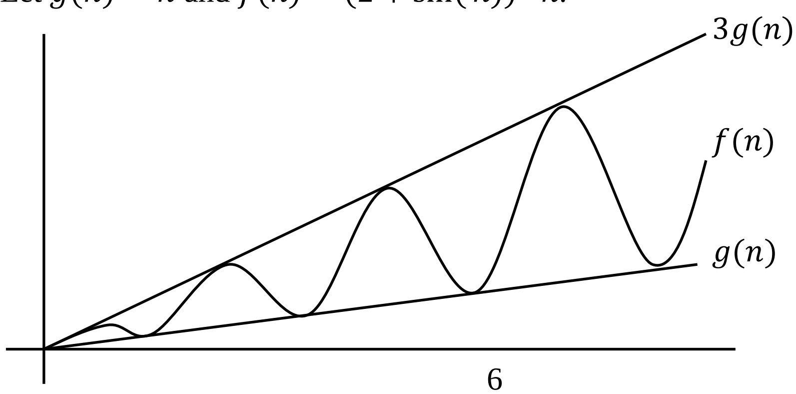
Although $o(g(n))$, $\omega(g(n))$, and a certain subset of $\Theta(g(n))$ are characterized by limits, the full sets $O(g(n))$, $\Omega(g(n))$, and $\Theta(g(n))$ have no such characterization as the following examples show.

Example A Let $g(n) = n$ and $f(n) = (1 + \sin(n)) \cdot n$.



Clearly $f(n) = O(g(n))$, but $\frac{f(n)}{g(n)} = 1 + \sin(n)$, whose limit does not exist, whence $f(n) \ne o(g(n))$. Observe also that $f(n) \ne \Omega(g(n))$ (why?). Therefore $f(n) \in O(g(n)) - \Theta(g(n)) - o(g(n))$, showing that the containment $o(g(n)) \subseteq O(g(n)) - \Theta(g(n))$ is in general strict.

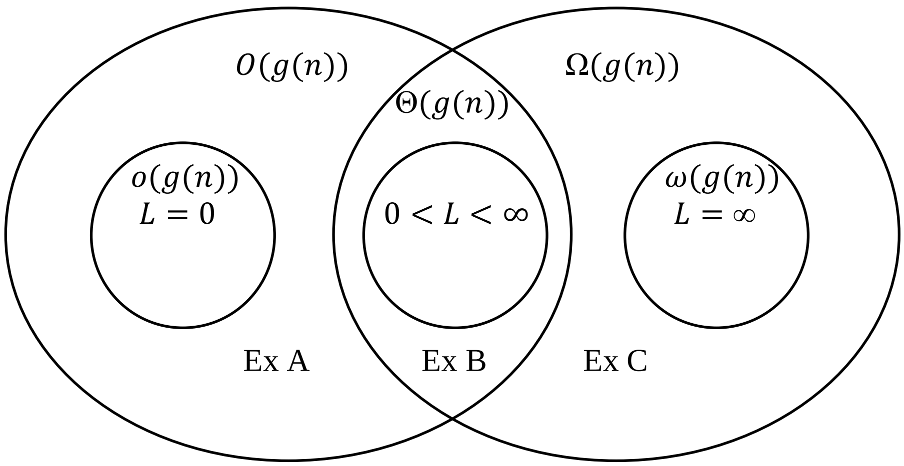
Example B Let $g(n) = n$ and $f(n) = (2 + \sin(n)) \cdot n$.



Since $n \leq (2+\sin(n)) \cdot n \leq 3n$ for all $n \geq 0$, we have $f(n) = \Theta(g(n))$, but $\frac{f(n)}{g(n)} = 2+\sin(n)$ whose limit does not exist.

Exercise 9 Find functions $f(n)$ and $g(n)$ such that $f(n) \in \Omega(g(n)) - \Theta(g(n))$, but $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ does not exist (even in the sense of being infinite), so that $f(n) \neq \omega(g(n))$. (Call this **Example C**.)

These limit theorems and counter-examples can be summarized in the following diagram. Here L denotes the limit $L = \lim_{n\to\infty} \frac{f(n)}{g(n)}$, if it exists.



In spite of the above counter-examples, the preceding limit theorems are a very useful tool for establishing asymptotic comparisons between functions. For instance recall the earlier exercise to show $(n+a)^b = \Theta(n^b)$ for real numbers a, and b with $b > 0$. The result follows immediately from the fact that $\lim_{n\to\infty} \frac{(n+a)^b}{n^b} = \lim_{n\to\infty} \left(1+\frac{a}{n}\right)^b = 1^b = 1$, since $0 < 1 < \infty$.

Exercise 10 Use limits to prove the following:

a. $n\ln(n) = o(n^2)$. More generally, show $n\log(n) = o(n^2)$ where the log has any base $b > 1$.
b. $n^5 2^n = \omega(n^{10})$.
c. If $P(n)$ is a polynomial of degree $k \geq 0$, then $P(n) = \Theta(n^k)$.
d. For any positive real numbers α and β: $n^\alpha = o(n^\beta)$ iff $\alpha < \beta$, $n^\alpha = \Theta(n^\beta)$ iff $\alpha = \beta$, and $n^\alpha = \omega(n^\beta)$ iff $\alpha > \beta$.
e. For any positive real numbers a and b: $a^n = o(b^n)$ iff $a < b$, $a^n = \Theta(b^n)$ iff $a = b$, and $a^n = \omega(b^n)$ iff $a > b$.
f. For any positive real numbers a and b: $\log_a(n) = \Theta(\log_b(n))$.
g. $f(n) + o(f(n)) = \Theta(f(n))$.

There is an analogy between the asymptotic comparison of functions $f(n)$ and $g(n)$, and the comparison of real numbers *x* and *y*.

$$
\begin{aligned}
f(n) = O(g(n)) &\quad \sim \quad x \le y \\
f(n) = \Theta(g(n)) &\quad \sim \quad x = y \\
f(n) = \Omega(g(n)) &\quad \sim \quad x \ge y \\
f(n) = o(g(n)) &\quad \sim \quad x < y \\
f(n) = \omega(g(n)) &\quad \sim \quad x > y
\end{aligned}
$$

If both *f* and *g* are polynomials of degrees *x* and *y* respectively, then the analogy is exact, as can be seen from parts (c) and (d) of the preceding exercise. In general though, the analogy is *not* exact since there exist pairs of functions which are not comparable.

Exercise 11 Let $f(n) = n^{\sin(n)}$ and $g(n) = \sqrt{n}$. Show that $f(n)$ and $g(n)$ are incomparable, i.e. $f(n)$ is in neither of the classes $O(g(n))$ nor $\Omega(g(n))$.

Exercise 12 Prove the following facts.

a. $\Theta(f(n)) + \Theta(g(n)) = \Theta(f(n) + g(n))$. In other words, if $h_1(n) = \Theta(f(n))$, and $h_2(n) = \Theta(g(n))$, then $h_1(n) + h_2(n) = \Theta(f(n) + g(n))$. Prove this also for O and Ω.

b. $\Theta(f(n)) \cdot \Theta(g(n)) = \Theta(f(n) \cdot g(n))$, using the same assumptions as in (a) above. Prove this also for O and Ω.

c. Suppose $f(n)$ and $g(n)$ are asymptotically positive. If $f(n) = \Theta\big(g(n)\big)$, then $1/f(n) = \Theta\big(1/g(n)\big)$.

d. Suppose $f(n) \ge \alpha$ for some $\alpha > 1$ and all sufficiently large *n*. Then $\lfloor f(n) \rfloor = \Theta(f(n))$, and $\lceil f(n) \rceil = \Theta(f(n))$. (Use that for any $x \in \mathbb{R}$: $x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x + 1$.) Note the existence of α is necessary for both conclusions, since $\lfloor 1/n \rfloor = 0$ and $\lceil 1/n \rceil = 1$, and yet neither 0 nor 1 are in $\Theta(1/n)$.

Exercise 13 Determine whether the first function is o, Θ, or ω of the second function.

a. n^n compared to $2^{n \lg n}$
b. n^n compared to $2^{n \ln n}$
c. 3^{2^n} compared to 2^{3^n}
d. $\sqrt{\ln n}$ compared to $\ln(\ln n)$.
e. $\sqrt{n}$ compared to $\sqrt{2}^{\ln n}$
f. $n^{\ln(\ln n)}$ compared to $2^{(\ln n)^2}$

Example 6 Let $a > 1$ and $b > 1$. Then

$$a^{b^n} = \begin{cases} \omega\left(b^{a^n}\right) & \text{if } a < b \\ \Theta(b^{a^n}) & \text{if } a = b \\ o(b^{a^n}) & \text{if } a > b \end{cases}$$

Proof:
We compute $\lim_{n\to\infty}\left(a^{b^n}/b^{a^n}\right)$ by first computing $\lim_{n\to\infty} \ln\,(a^{b^n}/b^{a^n})$. Observe by Exercise 9e above,

$$\ln\left(a^{b^n}/b^{a^n}\right) = b^n(\ln a) - a^n(\ln b) \to \begin{cases} \infty & \text{if } a < b \\ 0 & \text{if } a = b \\ -\infty & \text{if } a > b \end{cases} \quad \text{as } n \to \infty.$$

Applying the exponential function (base *e*) to these limits gives us

$$\lim_{n\to\infty}\left(a^{b^n}/b^{a^n}\right) = \begin{cases} \infty & \text{if } a < b \\ 1 & \text{if } a = b \\ 0 & \text{if } a > b \end{cases}$$

from which the result follows. ■

Definition 6 We say $f(n)$ is *asymptotically equivalent* to $g(n)$, and write $f(n) \sim g(n)$, if and only if $\lim_{n\to\infty}\left(f(n)/g(n)\right) = 1$.

Obviously $f(n) \sim g(n)$ implies that $f(n) = \Theta\left(g(n)\right)$, but it contains more information in that it tells us that the constant burried in the Θ notation is 1.

Exercise 14 Prove $f(n) \sim g(n)$ if and only if $f(n) = g(n) + o\left(g(n)\right)$.