

# Intermediate Meteorology

## Unit 2

### Reading Guide and Questions

Read Chapter 3

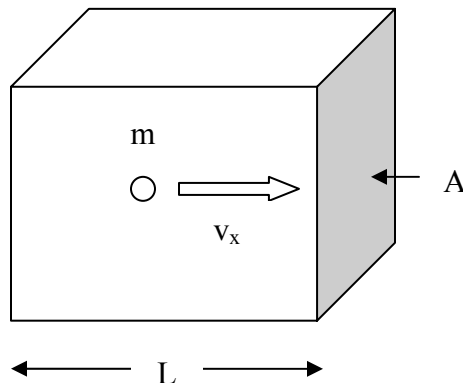
Ideal Gas Law results from a combination of Boyle's Law, Charles' Law and Avogadro's Hypothesis.

Avogadro's Number  $N_A = 6.0221415 \times 10^{23} \text{ mole}^{-1}$  is the number of molecules in one mole of gas. A mole of gas when multiplied by its molecular weight would equal the weight in grams.

Avogadro's Hypothesis states what in words?

Charles' Law states what in words?

Boyle's Law states what in words?



$$F = \frac{d(mv)}{dt} \rightarrow F_{ave} = \frac{mN\overline{v_x^2}}{L},$$

where  $N$  is the number of molecules and since the time is  $\frac{2L}{v_x}$  and the

change in velocity of an elastic collision is  $2v_x$ . Since the force is on one wall which is one third of the total pairs of wall (and each total area in a coordinate direction is twice the area of a face), the pressure on the walls is

$$p = \frac{mN\overline{v^2}}{3LA} = \frac{mN\overline{v^2}}{3V} = \frac{2N}{3V} \left( \frac{1}{2} \overline{mv^2} \right).$$

The kinetic energy of the molecules is  $\frac{3}{2}kT$  where  $k$  is the Boltzmann

constant and is equal to  $\frac{R^*}{N_A}$ . This is where the Ideal Gas Law gets its

form of  $pV = NkT$  or  $pV = nR^*T$  for  $n$  moles of gas. Since  $n = \frac{m}{M}$  we can

write the forms of the Ideal Gas Law that will serve our purposes most frequently  $p = \rho RT$  or  $p\alpha = RT$  for a unit mass. These forms, however, require different gas constants for different gases if one is to actually solve the equation.  $R$  will then be the Universal Gas Constant,  $R^*$ , divided by the molecular weight of the gas in question.

Using Table 1.1 and considering only the top three gases, calculate the apparent molecular weight of dry air.

$R_d = \underline{\hspace{2cm}}$  ;  $R_v = \underline{\hspace{2cm}}$  ;  $\epsilon = \underline{\hspace{2cm}}$  because  $\underline{\hspace{2cm}}$ .

Dalton's Law - 1802 ( $p_{total} = \sum_{i=1}^n p_i$ ) states what in words?

What is  $e$ ?

### Virtual Temperature

Since the gas constant is different for each gas, the presence of water vapor makes manipulation difficult. Therefore we shall change the value of temperature to account for the variation of vapor. A higher percentage of vapor makes air less dense. Therefore the virtual temperature is the temperature dry air would have to be in order to obtain the same density.

$$\frac{m_d + m_v}{V} = \rho = \rho'_d + \rho'_v \text{ where } \rho'_d \equiv \text{the density if dry air alone occupied the volume.}$$

Likewise for  $\rho'_v$ .

$$e = R_v \rho_v T \rightarrow \rho_v = \frac{e}{R_v T}$$

$$p_d = R_d \rho_d T \rightarrow \rho_d = \frac{p_d}{R_d T}$$

From Dalton's Law  $p = p_T - e$

$$\rho = \frac{e}{R_v T} + \frac{p - e}{R_d T}$$

$$\rho = \frac{p}{R_d T} \left( 1 - \frac{e}{p} + \frac{e R_d T}{R_v T p} \right)$$

$$\rho = \frac{p}{R_d T} \left( 1 - \frac{e}{p} + e \frac{\varepsilon}{p} \right)$$

$$\rho = \frac{p}{R_d T} \left( 1 - \frac{e}{p} (1 - \varepsilon) \right)$$

$$p = R_d \rho T_v \text{ where } T_v = \frac{T}{1 - \left( \frac{e}{p} \right) (1 - \varepsilon)}$$

Given:  $e=6\text{mb}$   $T=298\text{K}$   $p=1020\text{mb}$   $500\text{mb}$  is @  $5.3\text{km}$   
What is the density of the air at  $3.5\text{km}$ ?

## Hydrostatic Equation

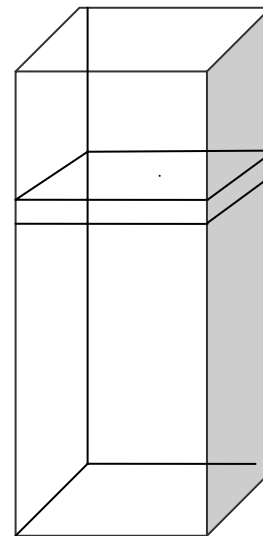
The mass of the slab of air per area is

$$\rho dz \cdot g = -dp$$

$$\frac{dp}{dz} = -\rho g \text{ or } g dz = -\alpha dp$$

$$\int_0^{\infty} \rho g dz = \int_{p(z)}^{P(\infty)} -dp$$

$$p(z) = \int_z^{\infty} g \rho dz$$



Hydrostatic Equilibrium – surfaces of constant pressure and constant density coincide and are horizontal throughout.

Archimedes Principle – a body immersed in a fluid experiences an upward force equal to the weight of the displaced fluid.

### Geopotential $\Phi$

$\Phi$  is the work that must be done against the earth's gravitational field in order to raise a unit mass from sea level to that point.

$$\Phi = \text{Force} \cdot \text{distance}$$

$$d\Phi = g dz (\text{per unit mass}) = -\alpha dp$$

$$\int_0^z d\Phi = \int_0^z g dz$$

$$\Phi(z) - \Phi(0) = \underline{\hspace{2cm}}$$

Why can't you take gravity outside of the integral?

Geopotential height (Z)

$$Z \equiv \frac{\Phi(z)}{g_0} = \frac{1}{g_0} \int_0^z g dz$$

$$\frac{dp}{dz} = -\rho g$$

Solve for  $g dz$  and use the ideal gas law to get to the following solution for thickness.

$$Z_2 - Z_1 = -\frac{R_d}{g_0} \int_{p_1}^{p_2} T_v \frac{dp}{p}$$

Take the integral of the above equation if the atmosphere is considered isothermal (or if an average temperature is assumed for the layer.)

Why is  $H \approx 29.3T$ ?

Equation 3.28 is a method of determining the mean virtual temperature. The virtual temperature is a function of height but can be integrated if a mean virtual temperature is assumed. The final form in equation 3.29 is the hypsometric equation. Write that here.

Using the hypsometric equation, explain how a cold core low and a warm core low differ.

Read section 3.2.4. There are some assumptions made in lowering pressure to sea level. Where would this be most problematic?

Read through problem 3.30. Equation 3.106 is used as the basis for the calibration of what instrument? Look that up on Wikipedia. Do you see that a variation in lapse rate from the standard atmosphere could cause problems with landing? Explain ways around this.

Do problems 3.19, 3.27

Assuming an isothermal atmosphere with a temperature of  $-33^{\circ}\text{C}$  and a surface pressure of 1000mb, estimate the levels at which pressure equals 100, 10 and 1 mb respectively.

### **1st Law of Thermodynamics**

The total energy of an object is the sum of its kinetic energy, its potential energy and its internal energy.

The kinetic energy is a function of the macroscopic velocity.  
The potential energy is a function of its (geopotential) height.  
The internal energy is a function of temperature (the internal kinetic energy and the potential energy of its configuration.)

A parcel of air will have some energy transferred to it ( $Q_{\text{in}}$ ) through thermal conduction or radiation. Note that convection does not count because there air in the box is the air in the box.

$q - w = u_2 - u_1$  where  $u_2$  is the final state of the gas and  $u_1$  is the initial state in terms of internal energy and is a function of those states only and not how it goes from one to the other.

The First Law can then be written as  $dq - dw = du$ . Eventually, all energy is written by random molecular motions.

How does equation 3.35 come about?

Notice that Fig. 3.4 is a type of thermodynamic diagram. In that diagram, what does the integral expressed in equation 3.36 express?

$dq = du + pd\alpha$  is a better way to express the First Law, but still not the best. Since  $du$  is a function of temperature, we want to have a better expression related to temperature than just  $du$ .

*Joule's Law* states that the internal energy of a gas is independent of its volume. Internal energy is a function of temperature and only temperature (of a gas.)

## Specific Heat

Specific heat is defined as the amount of heat required to raise a unit mass (1kg) of some substance 1K. Therefore, we can refine our understanding of specific heat by understanding it is the ratio of heat added to a substance to its corresponding temperature change. However, according to the First Law of Thermodynamics, if one adds heat,  $dq$ , some of the heat can be added to do work. It will therefore be important to put additional constraints on the state of the gas.

Specific heat at constant volume implies that the heat added results only in a temperature change (increase in internal energy) since zero expansion implies no work.

$$C_V \equiv \left( \frac{dq}{dT} \right)_{\alpha=const} = \left( \frac{du}{dT} \right)_{\alpha=const} \quad \text{Explain why this is true?}$$

Since Joule's Law states that internal energy is independent of volume, the change in internal energy is not constrained by a change in volume anyway.

$$\therefore C_V = \frac{du}{dT} \text{ or } du = C_V dT$$

A very important form of the First Law then becomes  $dq = C_V dT + pd\alpha$ .

$C_p = \left( \frac{dq}{dT} \right)_{p=const}$  If we constrain pressure to remain constant then heat added will result in work being done,  $dW$ , because of expansion.

Keep in mind that  $d(p\alpha) = pd\alpha + \alpha dp$  or  $pd\alpha = d(p\alpha) - \alpha dp$ .

$$dq = C_v dT + p d\alpha$$

$$dq = \underline{\hspace{2cm}}$$

$$dq = C_v dT + d(RT) - \alpha dp \text{ because } \underline{\hspace{2cm}}$$

$$dq = C_v dT + R dT + T dR - \alpha dp$$

$$dq = (\underline{\hspace{2cm}}) dT - \alpha dp$$

What happens to the last equation above if pressure is held constant?

$$dq = (C_v + R) dT \text{ or } \frac{dq}{dT} = C_v + R = C_p$$

Another important form of the First Law of Thermodynamics is

$dq = C_p dT - \alpha dp$ . A way to remember the two forms is to think that if p or V is kept constant in the specific heat term, then it must vary in the work term.

For dry air:

$$C_v = 717 J \cdot \text{deg}^{-1} \cdot \text{kg}^{-1}$$

$$R = 287 J \cdot \text{deg}^{-1} \cdot \text{kg}^{-1}$$

$$+ \underline{\hspace{2cm}}$$

$$C_p = 1004 J \cdot \text{deg}^{-1} \cdot \text{kg}^{-1}$$

For a monatomic gas -  $C_p: C_v: R = 5: 3: 2$

For a diatomic gas -  $C_p: C_v: R = 7: 3: 2$

## Enthalpy

$$dq = du + p d\alpha = (u_2 - u_1) + p(\alpha_2 - \alpha_1) = (u_2 + p\alpha_2) - (u_1 + p\alpha_1)$$

There is an final and initial state which we define as enthalpy.

$$h \equiv u + p\alpha \text{ so that } dq = h_2 - h_1$$

Now we differentiate the definition of enthalpy.

$$dh = du + d(p\alpha)$$

$$\text{from } dq = C_v dT + d(p\alpha) - \alpha dp$$

$$dq = dh - \alpha dp$$

Compare an alternate form of the first law,  $dq = C_p dT - \alpha dp$ .

$$\therefore dh = C_p dT \text{ or } h = C_p T$$

Remember that  $-\alpha dp = g dz = d\Phi$ .

$$dq = dh + d\Phi = d(h + \Phi)$$

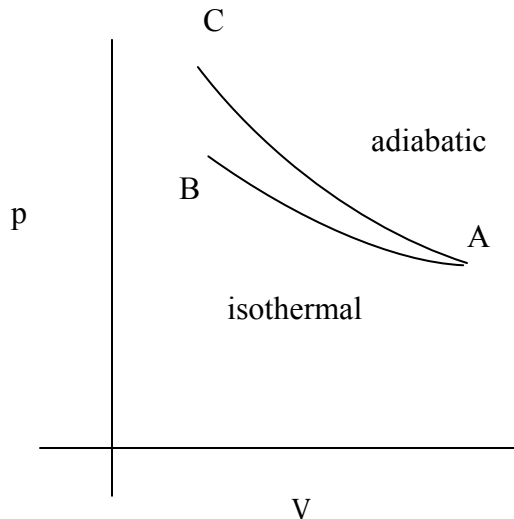
$$dq = d(C_p T + \Phi)$$

In an adiabatic process  $dq=0$ . This means that  $(C_p T + \Phi)$  is constant.

I hope you can see that in adiabatic ascent (where height increases) there is going to be a drop in the temperature.

Remember, in adiabatic processes,  $dq=0$ .

In isothermal process,  $dT=0$  and therefore  $du=0$ .



@A  $p\alpha = RT$

@C  $p_a \alpha_a = RT_a$  where a means adiabatic

@B  $p_i \alpha_i = RT_i$  where i means isothermal



We start at point A and compress a gas either isothermally or adiabatically. Comparing  $\frac{p_a \alpha_a}{p_i \alpha_i} = \frac{RT_a}{RT_i}$ .  $\alpha_a = \alpha_i$  since we need to compare when the gas is compressed to the same volume. Therefore,

$$\frac{p_a}{p_i} = \frac{T_a}{T_i}.$$

Why must the value of the right hand side must be greater than 1?

What does that say about the relation of the pressure at points B and C? Can you express in words why this is to be expected? Note that the graph expresses this in the way it is drawn.

Our parcel of air when moved adiabatically must be thermally insulated ( $dq=0$ ), must be hydrostatically balanced (internal pressure adjusts to external pressure) and total energy is a result of its internal energy and the parcel's potential energy (the parcel's kinetic energy is negligible.)

### Poisson's Equation

Potential temperature,  $\theta$ , is defined as the temperature a parcel of air would be if brought adiabatically to some reference pressure, most commonly 1000mb (1000hPa.)

$$dq = C_p dT - \alpha dp$$


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$$C_p dT - RT \frac{dp}{p} = 0$$


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$$\frac{C_p}{R} \frac{dT}{T} = \frac{dp}{p}$$


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$$\frac{C_p}{R} \int_{\theta}^T \frac{dT}{T} = \int_{p_0}^p \frac{dp}{p}$$


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$$\frac{C_p}{R} \ln\left(\frac{T}{\theta}\right) = \ln \frac{p}{p_0}$$


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$$\ln\left(\frac{T}{\theta}\right)^{C_p/R} = \ln \frac{p}{p_0}$$


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$$\left(\frac{T}{\theta}\right)^{C_p/R} = \frac{p}{p_0}$$


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$$\frac{T}{\theta} = \left(\frac{p}{p_0}\right)^{R/C_p}$$


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$$\theta = T \left(\frac{p_0}{p}\right)^{R/C_p}$$


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$$\theta = T \left(\frac{p_0}{p}\right)^{.286}$$


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What is  $\theta$  if  $T=230\text{K}$  at a pressure of  $400\text{mb}$ ? What would the temperature be at  $600\text{mb}$ ?

### Latent Heat

When some substance changes phase, heat ( $dQ$ ) is added but results in a zero temperature change. Latent heat of melting (fusion) is the heat required to convert a unit mass of a material from solid to liquid (or vice versa.) Latent heat of vaporization (condensation) is the heat required to convert a unit mass of a material from liquid to vapor (or vice versa.)

Latent heat of fusion of water is \_\_\_\_\_.

Latent heat of vaporization of water at  $0^\circ$  is \_\_\_\_\_.

The average kinetic energy of the molecules does not change. Therefore, there must be a point where the molecules reach a higher state of energy, going through a change in state. Those that are left have less energy, thus the heat added goes to the escaping molecules, not raising the average kinetic energy of the remaining ones. However, why does the air cool?

The answer is in the specific heats of air vs. water. The air has a certain amount of internal energy. Specific heat of water vapor at constant volume is  $1463 \text{ J} \cdot \text{deg} \cdot \text{kg}^{-1}$ . That means that if water vapor is in the air, the average specific heat is greater. If the specific heat is greater. The temperature will drop because in order to maintain that same temperature, heat would have to be added to the air to maintain the same temperature. But since it is not, the temperature decreases

because the average kinetic energy to maintain a constant temperature is insufficient.

### Virtual Temperature

What is  $T_v$  if  $T=30\text{ }^\circ\text{C}$  and  $w=20\text{ g}\cdot\text{kg}^{-1}$ ?

### Saturation Vapor Pressures

$e_s$  – saturation vapor pressure over a plane surface of water

$e_{si}$  – saturation vapor pressure over a plane ice surface

$$w_s = \frac{m_{vs}}{m_d} = \frac{\rho'_{vs}}{m'_d} = \frac{e_s / R_v T}{\left( p - e_s / R_d T \right)}$$

### Entropy

Do problems

3.19, 3.20, 3.27, 3.28, 3.32 3.33, 3.41, 3.58, 3.59, 3.61, 3.62